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NUMBER THEORY

★**Gupta, Hansraj. Symmetric functions in the theory of integral numbers.** Lucknow University Studies, no. 14. Allahabad Law Journal Press, Allahabad, India, 1940. vii+105 pp.

The main topic of this book is the function $G(n, r)$ which denotes the sum of the products taken r at a time of the numbers $1, 2, 3, \cdots, n$. The first two chapters give as preliminary material a score of elementary and fundamental theorems in the theory of numbers together with a number of exercises suitable for a beginner in number theory. Chapter III is devoted to the algebraic theory of the function $G(n, r)$; many expressions for this function are obtained in terms of factorials, Bernoulli numbers, differences of 0, etc. The final chapter deals with congruence properties of $G(n, r)$ and also symmetric functions of what are called "self-contained balanced" sets. A set of integers a_i is called self-contained modulo m if the numbers ta_i, taken modulo m, also belong to the set, where t is any number prime to m. The set is a balanced n set in case $n-a_i$ belongs to the set. Such sets are sufficiently like the integers $1, 2, 3, \cdots$ so that symmetric functions of them have congruence properties similar to those of $G(n, r)$. From these properties the theorems of Lagrange, Wilson, Fermat, Gauss, Wolstenholme, Leudesdorf, von Staudt and Bauer follow as special cases. There is a short bibliography at the end.

D. H. Lehmer (Berkeley, Calif.).

★**Robinson, Raphael M. Stencils for solving $x^2 \equiv a \pmod m$.** University of California Press, Berkeley, Calif., 1940. 14 pp.+274 stencils. $2.00.

This set of stencils is a mechanical method of applying Gauss's "method of exclusion" to the quadratic congruence (1) $x^2 \equiv a \pmod m$. As usual one excludes values of y in the equivalent equation (2) $x^2 = a + my$ by use of various small moduli E which are in this case $E = 5, 7, 9, 11, 13, 16, 17, 19, 23$. The stencils are actually Hollerith cards with 10 rows and 75 available columns. Each stencil (determined by E, a and m) is punched to show the values of $y < 750$ not excluded by E. By selecting and superposing the 9 stencils appropriate to a given congruence (1) and corresponding to the 9 excluding numbers E given above, one obtains at once the very few values of $y < 750$ which have a chance of satisfying (2) for some x. Actually trying these y's in (2) and solving for x, when possible, one obtains the desired solutions of (1). The method gives all solutions $x < m/2$ in case $m < 3000$. For larger values of m one may use the stencils several times, in fact n times in case $m < 750(n+1)^2$, by trying to solve the n congruences

$$z^2 \equiv k^2 a \pmod m, \qquad k = 1, 2, 3, \cdots, n,$$

from which $x \equiv z/k \pmod m$ is a solution of (1). Pages 6, 7 of the accompanying booklet contain 9 short tables enabling one easily to select the 9 stencils appropriate to a given quadratic congruence. *D. H. Lehmer* (Berkeley, Calif.).

Rigge, Olov. On a diophantine problem. Ark. Mat. Astr. Fys. **27A**, no. 3, 10 pp. (1940). [MF 3494]

The main result of this paper is as follows. If q is a prime and c an integer which is only divisible by primes not greater than $n/2$, then the diophantine equation

$$c(x+1)(x+2)\cdots(x+n) = y^q$$

is, for $q = 5$, $n = 50q^3$, solvable in integers x, y only when $y = 0$. Essentially this says that, if $q \geqq 5$ is a prime and x and n are integers for which $x \geqq n \geqq 50q^3$, then there is at least one number in the set $x+1, x+2, \cdots, x+n$ which is divisible by a prime $\geqq n$ to a power whose exponent is not divisible by q. The proof of this theorem is extremely involved and rests on a result of Thue concerning equations of the form $ax^r - by^r = f$. The author confines himself to the case $q \geqq 17$, where the proof is actually given, while the result for $q = 5, 7, 11, 13$ is merely stated. The paper is closely connected with one previously published by Erdös [J. London Math. Soc. **14**, 245–249 (1939); cf. these Rev. **1**, 39]. *I. A. Barnett* (Cincinnati, Ohio).

Aucoin, A. A. Diophantine equations of degree n. Bull. Amer. Math. Soc. **46**, 334–339 (1940). [MF 1844]

This is a generalization of a paper by the author and W. V. Parker on "Cubic Diophantine Equations" in the Nat. Math. Mag. **13**, 115–117 (1938). The present paper has to do with the integral solutions of the equation $f(x_1, \cdots, x_p) = g(y_1, \cdots, y_q)$ when f and g are homogeneous polynomials with integral coefficients of degrees n and m, and such that there exist a set of integers $a_1, \cdots, a_p$ not all zero for which the partial derivatives of f of all orders less than $n-1$ vanish when $x_i = a_i$. The discussion is incomplete since the condition (2) on page 334 is frequently not satisfied and the case $\sum_{j=1}^{p} a_j \partial f/\partial x_j \equiv 0$ has not been considered. In the corollary on page 338 the phrase "is given by" should be changed to "an equivalent solution may be obtained from." *I. A. Barnett* (Cincinnati, Ohio).

Vijayaraghavan, T. The general rational solution of some Diophantine equations of the form $\sum_{r=1}^{k+1} A_r x_r^{n_r} = 0$. Proc. Indian Acad. Sci., Sect. A. **12**, 284–289 (1940). [MF 3254]

Let i range over $1, \cdots, k$; r, s over $1, \cdots, k+1$. The general solution in rational nonzero x_r of $\sum A_i x_i^{n_i} = x_{k+1}^{n_{k+1}}$, where n_{k+1} is prime to $n_1 \cdots n_k$ ($\neq 0$), is given by $x_r = [X_r \Pi X_s^{M_s}]^{1/m_r}$, where the X_i are arbitrary nonzero l_ith powers of rational numbers, and $X_{k+1} = \sum A_i X_i$. Here $\alpha_0 = (n_1, \cdots, n_k)$, $\alpha_0 \alpha_i = (n_1, \cdots, n_{i-1}, n_{i+1}, \cdots, n_k)$, $\alpha_{k+1} = \alpha_0 \cdots \alpha_k$, $l_r = \alpha_{k+1}/\alpha_r$, $m_r = n_r/l_r$, $N = m_1 \cdots m_{k+1}$, N_r are integers such that $N_r \equiv -1 \pmod{m_r}$, $N_r \equiv 0 \pmod{\alpha_r N/m_r}$; $1 + \sum N_r = aN$, b and c satisfy $b\alpha_1 + c\alpha_2 = 1 - a$; $M_1 = N_1 + b\alpha_1 N$, $M_2 = N_2 + c\alpha_2 N$, $M_j = N_j$ ($j = 3, \cdots, k+1$). L. Tchacaloff and C. Karanicoloff [C. R. Acad. Sci. Paris **210**, 281–283 (1940); these Rev. **1**, 200] treated the case $k = 2$ with n_r coprime in pairs. *G. Pall* (Princeton, N. J.).

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Hua, Loo-Keng. On a system of Diophantine equations. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 312–313 (1940). [MF 3243]

An improvement is announced of a result of C. Mardjanichvili [C. R. (Doklady) Acad. Sci. URSS (N.S.) **22**, 467 (1939)] on the number of solutions of the system $x_1+\cdots+x_s=N_1, \cdots, x_1^n+\cdots+x_s^n=N_n$ in positive integers x_i; this result is said to hold for $s\geqq s_0(n)$, where $s_0(n)$ is an expression asymptotic to $7n^2 \log n$. *G. Pall.*

Sambasiva Rao, K. A correction. J. Indian Math. Soc. (N.S.) **4**, 125 (1940). [MF 3275]

[Concerning the author's paper in the same journal **3**, 262–265 (1939); these Rev. **1**, 135.] The proof of Theorem 1 in the author's paper is made rigorous by making two changes in the original argument. *R. D. James.*

Pillai, S. S. On Waring's problem $g(6)=73$. Proc. Indian Acad. Sci., Sect. A. **12**, 30–40 (1940). [MF 2750]

The main part of this paper is devoted to proving that every integer $N>e^{2400}$ is a sum of at most 73 integral sixth powers. The surprisingly small lower bound is obtained by the use of an inequality due to van der Corput [Nederl. Akad. Wetensch., Proc. **42**, 461–467 (1939)] which does not involve the function $d(n)$, the number of divisors of n. Not all the numerical computations are given in detail and in one instance at least (Lemma 4) the result seems to be incorrect. However, the final result will probably not be affected. To show that all integers not greater than e^{2400} are sums of at most 73 integral sixth powers, use is made of R. C. Shook's table of sixth powers and Dickson's method of ascent [Bull. Amer. Math. Soc. **39**, 701–727 (1933)]. Finally the integer 703 requires exactly 73 integral sixth powers in its representation. *R. D. James* (Saskatoon, Sask.).

Brauer, Alfred. On the non-existence of the Euclidean algorithm in certain quadratic number fields. Amer. J. Math. **62**, 697–716 (1940). [MF 2871]

A quadratic field is called Euclidean if the Euclidean division algorithm exists for any two integers in the field. The real fields $R(\sqrt{m})$ with $m=2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57$ are known to be Euclidean and it is an unsolved question whether there are any more [O. Perron, Math. Ann. **107**, 489–495 (1932); A. Oppenheim, Math. Ann. **109**, 349–352 (1934); R. Remak, Jber. Deutsch. Math. Verein. **44**, 238–250 (1934); E. Berg, Kgl. Fysiogr. Sällsk. Lund Förh. **5**, no. 5 (1935); N. Hofreiter, Monatsh. Math. Phys. **42**, 397–400 (1935)]. It has been shown that any additional (real) Euclidean fields must be of one of the forms: (I) $m=p\equiv 13 \pmod 8$, (II) $m=p\equiv 1 \pmod 8$, (III) $m=pq$, $p\equiv q\equiv 3, 7 \pmod 8$; p, q primes [H. Behrbohm and L. Rédei, J. Reine Angew. Math. **174**, 192–205 (1936); Hofreiter, l.c.]. Analytical methods were then used to prove that each of these cases yields at most a finite number of Euclidean fields [I, II by P. Erdös and Ch. Ko, J. London Math. Soc. **13**, 3–8 (1938); III by H. Heilbronn, Proc. Cambridge Philos. Soc. **34**, 521–526 (1938)].

In the present paper case I is treated in a purely elementary manner and it is proved that such a field cannot be Euclidean if $m>109$; this leaves only the values 61, 109 undecided here. The proof makes use of lemmas due to the above mentioned writers and also uses certain inequalities (due to the author) for the least odd quadratic non-residue u modulo a given prime p; for example, one of these inequalities states that $u<\sqrt{p}+4+2$ if $p\equiv 5 \pmod 8$. It is quite easy to prove that $R(\sqrt{p})$, $p\equiv 13 \pmod{24}$, is not Euclidean if $24[2(4p)^{3/5}+2(4p)^{1/5}+1]^2<p$, and this is certainly true if p is large enough. The limit thus found for p is lowered by improving the inequalities involved and the remaining cases are then treated individually.

H. W. Brinkmann (Swarthmore, Pa.).

Carlitz, L. An analogue of the Staudt-Clausen theorem. Duke Math. J. **7**, 62–67 (1940). [MF 3389]

In earlier papers [Duke Math. J. **1**, 137–168 (1935); **3**, 503–517 (1937)] the author has defined certain rational functions B_m (m is a multiple of p^n-1) in an indeterminate x with coefficients in a finite field $G(p^n)$ that are analogous to Bernoulli numbers. The following relation, for example, brings out this analogy:

$$\sum \frac{1}{E^m}=\frac{B_m}{g(m)}\cdot\xi^m;$$

here the summation is over all primary polynomials E in $G(p^n)$, ξ is a certain limiting expression and $g(m)$ is a "normalizing" factor analogous to $m!$ in the ordinary case. In the second of his earlier papers the author had proved an analogue to the von Staudt-Clausen theorem for the Bernoulli numbers. In the present paper he materially simplifies both the statement of the theorem and its proof.

H. W. Brinkmann (Swarthmore, Pa.).

Brown, D. M. Arithmetics of rational generalized quaternion algebras. Bull. Amer. Math. Soc. **46**, 899–908 (1940). [MF 3446]

A generalized quaternion algebra is a linear associative algebra of order four and rank two over the rationals with a basis $1=i, j, k, l$ satisfying $j^2=\alpha, k^2=\beta, jk=-kj=l$. In this paper the arithmetics of such an algebra in the sense of Dickson are obtained for the case when α, β are square-free rational integers by a method which simplifies and extends earlier treatments by Latimer and others. The parities of α, β are unrestricted; the bases of the arithmetics are obtained non-tentatively; the number of arithmetics for given α and β is determined. *M. Ward* (Pasadena, Calif.).

Tornheim, Leonard. Integral sets of quaternion algebras over a function field. Trans. Amer. Math. Soc. **48**, 436–450 (1940). [MF 3166]

The author investigates the arithmetic in quaternion algebras Q over function fields $F(z)$. Depending on the algebraic structure of F (all real numbers or a perfect field of characteristic 2) the theory of maximal integral sets S (relative to $F[z]$) is developed. In case that F is the field of real numbers it is shown that the fundamental number d of an algebra Q is the product of linear real polynomials and there is a single set S with the normalized base $[1, i, j, ij; i^2=-1, j^2=(\pm 1)d]$. Similar bases are constructed for other fields of coefficients. These bases are used as the tool for determining whether a generalized Euclidean algorithm is true for a set S and the discussion of the ideal theory in a maximal set. *O. F. G. Schilling* (Chicago, Ill.).

Vandiver, H. S. On improperly irregular cyclotomic fields. Proc. Nat. Acad. Sci. U. S. A. **27**, 77–83 (1941). [MF 3658]

The cyclotomic field $k(\zeta)$, where ζ is a primitive lth root of unity, is called irregular if the class number of the field is divisible by the prime l. An irregular field is called properly irregular if the "second factor" of the class number is

not divisible by l, improperly irregular in the contrary case. No improperly irregular fields are known and it is also known that for $l \leqq 617$ no field is improperly irregular. In this paper the author continues the study of such fields in the hope that he will eventually reach a contradiction and thus show that none exists. He distinguishes two kinds of improperly irregular fields. The first type contains no unit congruent to a rational integer, modulo $(1-\zeta)^{l-1}$, and has properties similar to those of regular fields. Thus it turns out that for any real integer ω prime to l we can find a unit η in $k(\zeta)$ such that $\omega\eta$ is congruent to a rational integer, modulo $(1-\zeta)^{l-1}$ [compare this to Hilbert, Werke, I, p. 288]. For a field of the second type the Bernoulli numbers B_{a+xy} are divisible by l^2; here a is a certain fixed number, $\mu=\frac{1}{2}(l-1)$ and $y \geqq 0$ is any integer.

H. W. Brinkmann (Swarthmore, Pa.).

Inaba, Eizi. Über die Struktur der l-Klassengruppe zyklischer Zahlkörper vom Primzahlgrad l. J. Fac. Sci. Imp. Univ. Tokyo. Sect. I. 4, 61–115 (1940). [MF 3067]

Let K be a cyclic field of prime degree l over an algebraic number field k. The l-class group ($\mathfrak{L}$) of K is the subgroup of the group of ideal classes of K that contains all the classes whose order is a power of l. The purpose of this paper is to show how the structure of $\mathfrak{L}$ can be determined in any given case by means of operations involving only numbers in k. To do this the author introduces a succession of subgroups $\mathfrak{R}_n$ of $\mathfrak{L}$. Thus $\mathfrak{R}_1$ is the subgroup consisting of the "two-sided" (ambige) classes in $\mathfrak{L}$, that is, the classes C in $\mathfrak{L}$ for which $C^{1-\sigma}=1$. Here σ is a generator of the Galois group of K over k and C^{σ} means the corresponding conjugate of C. In general $\mathfrak{R}_n$ is the subgroup consisting of the classes C in $\mathfrak{L}$ for which $C^{(1-\sigma)^n}=1$. It is shown first that $\mathfrak{L}$ can be expressed as the direct product of "cyclic" groups, that is, groups that consist of the symbolic powers of a single generating element. The symbolic "orders" of these cyclic groups are of the form $(1-\sigma)^n$ and the totality of these forms the set of "invariants" of $\mathfrak{L}$. It then turns out that the index of $\mathfrak{R}_{n-1}$ in $\mathfrak{R}_n$ is l^{s_n}, where s_n is the number of "invariants" divisible by a given $(1-\sigma)^n$.

The essential step is now to show that $\mathfrak{R}_n/\mathfrak{R}_{n-1}$ is isomorphic to a certain group of numbers in k. Let $\mathfrak{G}_n$ be the group of ideals generated by a set of representative ideals chosen from the basis classes of $\mathfrak{R}_n/\mathfrak{R}_{n-1}$. Let G_n be the group of norms of the ideals in $\mathfrak{G}_n$ and H_n the group of norms of those ideals in $\mathfrak{G}_n$ that are also in $\mathfrak{R}_{n-1}$. Then $\mathfrak{R}_n/\mathfrak{R}_{n-1}$ is shown to be isomorphic to G_n/H_n. All the norms mentioned are principal ideals in k and the study of $\mathfrak{R}_n/\mathfrak{R}_{n-1}$ is thus reduced to that of these groups of numbers in k. The author shows next how to construct these groups of numbers step by step. The start is made by determining the classes containing two-sided ideals, and these are found from the relative discriminant of K/k. A process for continuing from one group to the next is then described. The numbers used for the purpose of this continuation can be described by means of certain Artin symbols (generalization of the Legendre symbol) and can be found by solving certain Diophantine equations. If k is the field of rational numbers, the structure of the group of two-sided classes is readily determined. The author illustrates how his constructions can be carried out by working through a numerical example ($l=3$). If, in addition, K is a quadratic field ($l=2$), the whole procedure becomes very much simpler. The classes of order $(1-\sigma)^n$ are then the classes of order 2^n and it is at once possible to determine, for each K, the number (e_n) of invariants of the class group that are divisible by 2^n. Beyond the classical result of Gauss, who determined e_1, e_n has been studied by Rédei and Reichardt [Rédei and Reichardt (e_2), J. Reine Angew. Math. 170, 69–74 (1933); Reichardt (e_n), ibid., 75–82; Rédei (e_2), J. Reine Angew. Math. 171, 55–60 (1934); Rédei (e_3), ibid., 131–148]. The author is able to deduce the theorem of Rédei and Reichardt about e_2 from his general results. Here also a number of numerical illustrations are given.

The discussion just given deals with the "absolute" class group, with the restriction that l be prime to the class number of k. The author also derives similar results for the l-class group with respect to any (generalized) modulus $\mathfrak{m}$ which is prime to the relative discriminant of K/k. In particular this takes care of the case of "restricted" equivalence for quadratic fields.

H. W. Brinkmann.

Niven, Ivan. Integers of quadratic fields as sums of squares. Trans. Amer. Math. Soc. 48, 405–417 (1940). [MF 3164]

The author proves that in an imaginary quadratic field every integer of the form $a+2b\sqrt{-m}$ is expressible as a sum of three squares of integers of the form $x+y\sqrt{-m}$ (x, y rational integers). To do this he derives a condition that such an integer be expressible as a sum of two squares. Thus for the case of Gaussian integers ($m=-1$), an integer of the form $a+2bi$ is the sum of two squares if and only if not both $a/2$ and b are odd integers; the theorem about three squares follows immediately. In the case where $m \equiv 3 \pmod 4$, it readily follows as a corollary that every integer in the field is expressible as the sum of three such squares. The case of a real quadratic field is also briefly treated. The main tool used is a theorem of Mordell on expressing a quadratic form as the sum of two squares of linear forms with integral coefficients [Math. Z. 35, 1–15 (1932)]. For example, in the Gaussian case, the author sets $a+2bi=(a+l)+2bi+li^2$ and then tries to choose t so as to make this quadratic form in $1, i$ the sum of two squares. By Mordell's theorem this turns out to be possible if and only if a, b satisfy the conditions stated above.

H. W. Brinkmann (Swarthmore, Pa.).

Wenkov, B. Über die Reduction positiver quadratischer Formen. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 4, 37–52 (1940). (Russian. German summary) [MF 1981]

The author introduces (after Minkowski) the space P of the coefficients a_{ij} ($i \leqq j$) of positive real quadratic forms $f=\sum_{i=1}^{n} a_{ij}x_ix_j$ and studies the problem of determination of the fundamental domains, with respect to P, of the group G_n of unimodular transformations of f. He considers the domain $V(\varphi)$ of points $f=(a_{11}, a_{12}, \cdots, a_{nn})$ defined by inequalities

$$(f, \Phi) \leqq (f, \mathfrak{G}'\Phi\mathfrak{G}) \qquad \text{for every } \mathfrak{G} \text{ in } G_n,$$

where Φ is the adjoint of (the form) f and (f, Φ) is the scalar product of f and Φ. Employing results of Minkowski and Voronoi [H. Minkowski, Diskontinuitätsbereich für arithmetische Äquivalenz, Gesammelte Abhandlungen, Bd. II, 1911; G. Voronoi, J. Reine Angew. Math. 133 (1908)], the author shows that $V(\varphi)$ is a pyramid with a finite number of faces, and that should φ have no automorphs then $V(\varphi)$ is a fundamental domain of G_n. In conclusion, the author points out that with the help of this more general fundamental domain one can see that for $n>2$ a fundamental

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domain may be deformed continuously. This is unlike the case for $n=2$, where the boundary of this domain consists entirely of forms with automorphs. *A. E. Ross.*

Humbert, Pierre. Théorie de la réduction des formes quadratiques définies positives dans un corps algébrique K fini. Comment. Math. Helv. **12**, 263–306 (1940). [MF 3058]

In this paper Minkowski's theory of the equivalence of positive definite quadratic forms with respect to transformations with integral coefficients (and determinant ± 1) is extended to transformations whose coefficients are integers in a finite algebraic field K and whose determinant is a unit in K. As it stands the group of such "unimodular" transformations is not properly discontinuous in the space of the coefficients of the form. The author therefore deals with a system S of forms consisting of a positive definite quadratic form with real coefficients for each real conjugate of K and a pair of conjugate imaginary Hermitian forms for each pair of conjugate imaginary conjugates of K. These forms in S are then simultaneously transformed by the transformations conjugate to the given unimodular transformation in K. In the space of the coefficients of S the group is then found to be properly discontinuous and results analogous to Minkowski's hold: There exists a fundamental domain formed by the union of several convex solid angles R_i each bounded by a finite number of hyperplanes; each R_i is in contact with a finite number of equivalent fundamental domains. In particular it follows that the unimodular group here considered has a finite number of generators, a fact that had been proved by Hurwitz [Gesammelte Werke, v. 2, p. 244].

The method used is essentially that of Minkowski and consists in finding a "reduced" system equivalent to S. The main modification is that the process of "reduction" is accomplished in two steps, this being made necessary by the fact that the class number of K may be greater than one. First a given system S is transformed by all non-singular transformations with integral coefficients. From the resulting family of systems, systems are then selected successively by minimum requirements similar to those used by Hermite and Minkowski. One or more systems S_1 thus result, each being obtained by transforming S by a transformation A_0 which will not usually be unimodular. It is shown, however, that the norm of the determinant of A_0 is bounded, and, as a consequence, that $A_0=UA_r$, where U is unimodular and A_r belongs to a certain fixed set of non-singular matrices A_i $(i=1, 2, \cdots, N)$. Transformation by A_r^{-1} then yields a system S_2 equivalent to S and this is called a "reduced" system. To each matrix A_i corresponds a domain R_i of reduced systems and the union of these is the fundamental domain of the unimodular group. The fundamental domain thus found is then shown to have the properties stated above. Throughout these proofs the main tool employed is Minkowski's theorem on linear forms.

H. W. Brinkmann (Swarthmore, Pa.).

Siegel, Carl Ludwig. Einheiten quadratischer Formen. Abh. Math. Sem. Hansischen Univ. **13**, 209–239 (1940). [MF 2382]

This paper contains an extended discussion of results presented in Hamburg in January, 1939. The first part of the paper is devoted to a simple development of Minkowski's theory of the reduction of positive quadratic forms [J. Reine Angew. Math. **129**, 220–274 (1905); Werke, Bd. 2, pp. 53–100. Cf. Bieberbach und Schur, S.-B. Preuss. Akad. Wiss., Phys.-Math. Kl. **1928**, 510–535] and of Hermite's theory of the reduction of indefinite quadratic forms [Hermite, J. Reine Angew. Math. **47**, 330–342 (1854); Oeuvres, T. 1, pp. 220–238; Stouff, Ann. École Norm. (3) **19**, 89–118 (1902)]. The second part contains the subject proper of this paper, that is, the study of the group Γ of integral automorphs of an indefinite quadratic form $\mathfrak{x}'\mathfrak{S}\mathfrak{x}$. Here the author proves that this group of automorphs has a finite number of generators (a deep and difficult theorem), thus generalizing the well-known result for indefinite binary quadratic forms. He also establishes the convergence of the integral

$$v(\mathfrak{S})=D^{-n/2}\int_G |\mathfrak{T}|^{-n/2}d\mathfrak{P}$$

(where G is the fundamental domain of Γ in a suitably chosen space $\mathfrak{P}$), and employs it to define the mass $\mu(\mathfrak{S})$ of Γ. [Cf. Siegel, Math. Z. **44**, 398–426 (1939).] He next defines the mass (weight) of a genus as the sum $\mu(\mathfrak{S}_1)+\cdots+\mu(\mathfrak{S}_n)$, where $\mathfrak{S}_i$ are representatives of distinct classes of that genus and gives a formula for the weight of the genus of indefinite quadratic forms [cf. Siegel, Ann. of Math. (2) **37**, 230–263 (1936)]. *A. E. Ross* (St. Louis, Mo.).

Skolem, Th. On functions of the form $\sum_{i=0}^{m-1} f_i(x)P_m(x-i)$, where all $f_i(x)$ are polynomials and $P_m(x)=1$ or 0 according as $x\equiv 0$ or $\not\equiv 0 \pmod m$. Norsk Mat. Tidsskr. **22**, 52–63 (1940). (Norwegian) [MF 3314]

The ring R_1 of all rational polynomials $f(x)$ which take on integral values for integral values of x does not have the property that for any two polynomials $f(x)$ and $g(x)$ in R_1 there is an $h(x)$ in R_1 such that for every integer x the value of $h(x)$ is the greatest common divisor of the values of $f(x)$ and $g(x)$. The author solves the problem of extending R_1 to a ring R_ω which has this property. A ring R_m is defined consisting of all functions of the form given in the title which take on integral values for integral values of x. The union of rings R_m for all integers m is a ring R_ω having the desired property. The author points out that although R_ω achieves closure in the above sense its arithmetic is complicated by the non-existence of primes.

H. T. Engstrom (New Haven, Conn.).

Mahler, Kurt. Über Polynome mit ganzen rationalen Koeffizienten. Mathematica, Zutphen. B. **8**, 173–182 (1940). [MF 1881]

Verfasser betrachtet das Polynom $f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n$ $(n\geqq 3,\ a_0\neq 0)$ und die zugehörige Binärform $F(x, y)=a_0x^n+a_1x^{n-1}y+\cdots+a_ny^n$. Durch Abschätzung der Resultante von F_x' und F_y' leitet Verfasser verschiedene Ungleichungen her, unter anderm eine Abschätzung nach unten von $|f(x)|$, die von Wichtigkeit ist bei der Bestimmung des Masses aller Mahlerschen S-Zahlen. [Für weitere Literatur vgl. diese Rev. **1**, 137.] *J. F. Koksma.*

Mahler, Kurt. On the product of two complex linear polynomials in two variables. J. London Math. Soc. **15**, 213–236 (1940). [MF 3141]

Let $\alpha, \beta, \gamma, \delta$ be four complex numbers such that $\alpha\delta-\beta\gamma=1$ and ξ, η two other complex numbers. Consider the product $P(x, y)=(\alpha x+\beta y+\xi)(\gamma x+\delta y+\eta)$, where x and y are variable integers in the Gaussian field $K(\sqrt{-1})$. Some time ago E. Hlawka [Monatsh. Math. Phys. **46**, 324–334 (1938)] proved that the minimum of the absolute value of $P(x, y)$ is not greater than $\frac{1}{2}$; he determined also the limiting

case, when the sign of equality is necessary. The author obtains a new proof using the theory of reduction of positive binary Hermitian forms and the solution of a certain simple geometrical extremum problem. Two analogous theorems for the quadratic fields $K(\sqrt{-2})$ and $K(\sqrt{-3})$ are proved by means of the same method. *C. L. Siegel.*

Hofreiter, Nikolaus. Über das Produkt von Linearformen. Monatsh. Math. Phys. **49**, 295–298 (1940). [MF 3093]

If d is the absolutely least discriminant for all algebraic fields of degree n with r real and $2s$ imaginary conjugates ($r+2s=n$), it is observed that there exist linear forms $L_1, \cdots, L_n$ of determinant 1 (r with real, and s pairs with conjugate complex, coefficients) in rational integer variables $x_1, \cdots, x_n$ such that $|L_1 \cdots L_n| < c$ has only the solution x_i all zero, if $c^2 \leqq 1/d$. This follows from the fact that $N(\omega_1 x_1 + \cdots + \omega_n x_n) \geqq 1$, being the norm of an algebraic integer. The special cases $n=2$, $r=2$; $n=2$, $r=0$; $n=3$, $r=3$; and $n=3$, $r=1$ give the results known to be best possible, $c^2 = 1/5$, $1/3$, $1/49$, $1/23$ [cf. H. Davenport, Proc. London Math. Soc. (2) **44**, 412–431 (1938), and **45**, 98–125 (1939)]. If $n=4$ the determination of d [J. Mayer, Akad. Wiss. Wien, S.-B. **138**, 733–742 (1929)] gives $c^2 = 1/725$, $1/275$ and $1/117$, according as $r=4$, 2 or 0.

G. Pall (Princeton, N. J.).

Hofreiter, Nikolaus. Diophantische Approximationen komplexer Zahlen. Monatsh. Math. Phys. **49**, 299–302 (1940). [MF 3094]

Set $\lambda=1$ if $m \not\equiv 3$, $\lambda=2$ if $m \equiv 3 \pmod 4$. If $\alpha_1, \cdots, \alpha_{n-1}$ are complex numbers, linearly independent with respect to the field $k(i\sqrt{m})$, there exist infinitely many integers $q_1, \cdots, q_n$ in this field, such that

$$|L| = |\alpha_1 q_1 + \cdots + \alpha_{n-1} q_{n-1} - q_n| < \frac{K_n}{|q_1 \cdots q_{n-1}|}.$$

Here $\lambda K_2 = \sqrt{(m/2)}$, $K_3^2 = 8m^{3/2}/(27\lambda^3\sqrt{3})$, $K_4 = m/(4\lambda^2)$, $K_n^2 < (n+1)!(2\sqrt{m}/(n\pi\lambda))^n$. The proof uses the inequality

$$n(|q_1|^2 \cdots |q_{n-1}|^2 |L|^2)^{1/n} \leqq \frac{|q_1|^2}{t} + \cdots + \frac{|q_{n-1}|^2}{t} + |L|^2 t^{n-1},$$

where $q_\nu = y_\nu + (\lambda - 1 + i\sqrt{m})z_\nu/\lambda$, and thus reduces the problem to one on minima of quadratic forms. *G. Pall.*

Robinson, Raphael M. The approximation of irrational numbers by fractions with odd or even terms. Duke Math. J. **7**, 354–359 (1940). [MF 3406]

The author studies the problem [A. Hurwitz, Math. Ann. **39**, 279–285 (1891)] of approximation to an irrational number ξ by rational numbers A/B subject to the condition of $|\xi - A/B| < 1/\mu B^2$ for various values of μ. The results obtained extend those of Hurwitz and of W. T. Scott [Bull. Amer. Math. Soc. **46**, 124–129 (1940); these Rev. **1**, 203] and are the best of their type. The methods, unlike those of Scott [cf. L. R. Ford, Proc. Edinburgh Math. Soc. **35** (1916)], involve the traditional use of continued fractions, their convergents and their secondary convergents. The author's bibliography contains only the paper of Scott.

W. Leighton (Houston, Tex.).

Veldkamp, G. R. Ein Transzendenz-Satz für p-adische Zahlen. J. London Math. Soc. **15**, 183–192 (1940). [MF 3136]

The well-known theorem of Gelfond and Schneider [C. R. (Doklady) Acad. Sci. URSS (N.S.) **2**, 1–6 (1934); J. Reine Angew. Math. **172**, 65–69 (1934)] concerning the transcendence of ω^θ for irrational algebraic θ and algebraic $\omega \neq 0, 1$ has an analogue in the field of p-adic numbers: Let θ and ω be p-adic algebraic integers, θ irrational, $\omega \not\equiv 1$ and $\omega \equiv 1 \pmod p$ and define the p-adic power ω^θ by the convergent series

$$\omega^\theta = e^{\theta \log \omega} = \sum_{k=0}^{\infty} \frac{(\theta \log \omega)^k}{k!}, \quad \log \omega = -\sum_{k=1}^{\infty} \frac{(1-\omega)^k}{k};$$

then ω^θ is transcendent. The proof uses the method of Th. Schneider and a lemma from a paper of K. Mahler [Compositio Math. **2**, 259–275 (1935)], where another proof of this theorem based on Gelfond's idea had already been given. *C. L. Siegel* (Princeton, N. J.).

Popken, J. On Lambert's proof for the irrationality of π. Nederl. Akad. Wetensch., Proc. **43**, 712–714 (1940). [MF 3108]

The author condenses Lambert's demonstration for the irrationality of π into a new and very short proof whose simplicity is surprising, but his initial formula must also appear very surprising to a reader who is not at all acquainted with Lambert's proof. *G. Pólya.*

Ingham, A. E. On two classical lattice point problems. Proc. Cambridge Philos. Soc. **36**, 131–138 (1940). [MF 1713]

The object of this note is to prove

$$\limsup \frac{P(x)}{x^{\frac{1}{4}}} = +\infty, \quad \liminf \frac{\Delta(x)}{x^{\frac{1}{4}}} = -\infty,$$

where $P(x)$ is the error term in the problem of the lattice points of a circle, and $\Delta(x)$ the error term in Dirichlet's divisor problem. *J. G. van der Corput* (Groningen).

Lehmer, D. H. The lattice points of an n-dimensional tetrahedron. Duke Math. J. **7**, 341–353 (1940). [MF 3405]

The author constructs polynomials which approximate to the number of lattice points lying inside or on the boundary of the n-dimensional tetrahedron bounded by the n coördinate hyperplanes $x_1=0$, $x_2=0$, $\cdots$, $x_n=0$ and the hyperplane

$$\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_n x_n = \lambda,$$

where ω_i are positive and λ is a non-negative parameter. Denoting the total number of such points by

$$N_n(\lambda) = N_n(\lambda | \omega_1, \omega_2, \cdots, \omega_n),$$

he constructs polynomials $P_n(\lambda | \omega_1, \omega_2, \cdots, \omega_n)$ and $Q_n(\lambda | \omega_1, \omega_2, \cdots, \omega_n)$ each of degree n in λ with coefficients depending on $\omega_1, \omega_2, \cdots, \omega_n$ and such that the inequalities

$$P_n(\lambda | \omega_1, \cdots, \omega_n) < N_n(\lambda | \omega_1, \cdots, \omega_n) < Q_n(\lambda | \omega_1, \cdots, \omega_n)$$

are valid for all $\lambda \geqq 0$. The $P_1(\lambda)$, $P_2(\lambda)$, $\cdots$, $P_n(\lambda)$ are obtained as successive solutions of a sequence of linear difference equations. More precisely, if

$$S_k(x, \omega) = \sum_{1 \leq \mu \leq x/\omega} (x - \mu\omega)^k,$$

it is shown that

$$(1) \quad \omega^k \{B_{k+1}(x/\omega) - M_{k+1}\} \leqq (k+1) S_k(x, \omega) \leqq \omega^k \{B_{k+1}(x/\omega) - m_{k+1}\},$$

where $B_\nu(x)$ is the νth Bernoulli polynomial and M_ν, m_ν are its maximum and minimum, respectively, in the interval $0 \leqq x \leqq 1$. Taking $P_1(\lambda/\omega) = \lambda/\omega$, and assuming that

$P_{k-1}(\lambda)$ has already been constructed, the formula (which is immediate)

$$N_k(\lambda|\omega_1, \cdots, \omega_k) = \sum_{0 \leqq \mu \leqq \lambda/\omega_k} N_{k-1}(\lambda - \mu\omega_k | \omega_1, \cdots, \omega_{k-1})$$

yields

$$(2)\quad N_k(\lambda) > \sum_{0 \leqq \mu \leqq \lambda/\omega_k} P_{k-1}(\lambda - \mu\omega_k) = P_{k-1}(\lambda) + \sum_{\nu=0}^{k-1} p_\nu^{(k-1)} S_\nu(\lambda, \omega_k).$$

Writing

$$K_n(t) = \begin{cases} tm_n/n & \text{if } t \leqq 0, \\ tM_n/n & \text{if } t \geqq 0, \end{cases}$$

and substituting from (1) into (2), we obtain

$$N_k(\lambda) > P_{k-1}(\lambda) + \sum_{\nu=0}^{k-1} p_\nu^{(k-1)} \frac{\omega_k^\nu}{\nu+1} B_{\nu+1}(\lambda/\omega_k) - \sum_{\nu=0}^{k-1} \omega_k^\nu K_{\nu+1}(p_\nu^{(k-1)}),$$

and the right member, a polynomial in λ of degree k, is taken for $P_k(\lambda)$. Expanding the Bernoulli polynomials and collecting the coefficients of the various powers of λ, a recursion formula is obtained for $p_\nu^{(\nu)}$. The polynomial $Q_n(\lambda)$ is constructed in a similar way.

It is shown among other things that for each $n>0$ infinitely many n-dimensional tetrahedra exist for which

$$N_n(\lambda) - P_n(\lambda) = O(\lambda^{n-\epsilon}),$$

and the author compares (for $n=5$, $\lambda=1, 2, 3, 5, 8, 10, 10.5$) his polynomials with approximating polynomials of other writers (one of which was used by J. B. Rosser, another by A. E. Western and a third by the reviewer).

D. C. Spencer (Cambridge, Mass.).

Chatrovsky, L. Sur les bases minimales de la suite des nombres naturels. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 335–340 (1940). (Russian. French summary) [MF 3325]

A sequence $\varphi = \{n_1, n_2, \cdots\}$ of integers n_k is called a basis of order h (of the natural numbers n) if every n is a sum of h numbers n_k. If $\varphi(x)$ is the number of n_k less than x, necessarily $\varphi(x) \geqq c_h x^{1/h}$. An elementary method is given of constructing infinitely many distinct minimal bases (that is, with $\varphi(x) = O(x^{1/h})$). A. Stöhr [Math. Z. **42**, 739–743 (1937)] and D. Raikov [Rec. Math. [Mat. Sbornik] N.S. **2 (44)**, 595 (1937)] had previously found an example.

G. Pall (Princeton, N. J.).

Hua, Loo-Keng. On a lemma due to Vinogradow. C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 419–420 (1939). [MF 2048]

Set

$$\phi = \left| \sum_{x=1}^{P} \exp\left(2\pi i(\alpha_n x^n + \cdots + \alpha_1 x)\right) \right|,$$

$$I_r = \int_0^1 \cdots \int_0^1 \phi^r \, d\alpha_1 \cdots d\alpha_n.$$

Vinogradow proved [Rec. Math. [Mat. Sbornik] N.S. **3 (45)**, 435–470 (1938)] that $I_r \ll P^{r-\frac{1}{2}n(n+1)}$, where r is the greatest even integer less than $Ln(n+1)(n+2)\log n$, and $L_2=4.81, \cdots, L_{13}=4.12$, $L_n=4.10$ if $n \geqq 14$. Hua states an improvement for $n \leqq 10$, namely $I_s < KP^{s-\frac{1}{2}n(n+1)+\epsilon}$, where $s = 2n + 2^t u + 2^{t+1}(t+2) + \cdots + 2^{n-1}n$, and t is the least integer such that $u = \frac{1}{2}(t+1)(t+2) - n \geqq 0$, and the constant K depends only on n and ϵ. He proves that for $2l > \frac{1}{2}n(n+1)$, $I_{2l} \gg P^{2l-\frac{1}{2}n(n+1)}$.

G. Pall (Princeton, N. J.).

Hua, Loo-Keng. On a theorem due to Vinogradow. Quart. J. Math., Oxford Ser. **11**, 161–176 (1940). [MF 3126]

Proof is given of results for $2 \leqq n \leqq 10$ slightly better than those announced in the paper reviewed above. *G. Pall.*

Segal, B. On certain sets of integers. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 319–334 (1940). (Russian. English summary) [MF 3324]

Let $n \geqq 2$ be an integer, $l \geqq r+1$, r the greatest even number less than $4 \cdot 81 n(n+1)(n+2)\log n$. Let $p_1, \cdots, p_l$ be given primes, and let $\sigma_1, \cdots, \sigma_l$ range over integer nth powers. Write $\nu = 1/n$, $\rho = (\log\log N)^{1-\epsilon}$, $\rho_0 = (\log\log N)^{1-\epsilon_0}$, $\beta_l = \log p_1 \cdots \log p_l$, $\gamma = (\Gamma(1+\nu))^l / \Gamma(l\nu)$. Then the number I_N of representations of N in the form $N = dp_1^{\sigma_1} \cdots p_l^{\sigma_l}$, with $e^{-\Delta} \leqq d \leqq e^{\Delta}$ and $\Delta = e^{-\rho}$, is given by

$$I_N = 2(\gamma/\beta_l^r)(\log N)^{l\nu-1}\Delta + O((\log N)^{l\nu-1} e^{-\rho_0}),$$

where $0 < \epsilon_0 < \epsilon$. The case where the σ_i range over all integers or all primes was considered in earlier articles [same Bull. **3**, 519–538 (1939); cf. these Rev. **2**, 40]. Estimates of sums due to I. M. Vinogradow and A. D. Gelfond are used. Lemmas 7 and 8 are alternative to, and are said to have wider applicability than, some estimates by van der Corput [Nederl. Akad. Wetensch., Proc. **42**, 461–467 (1939)].

G. Pall (Princeton, N. J.).

Rosser, Barkley. Explicit bounds for some functions of prime numbers. Amer. J. Math. **63**, 211–232 (1941). [MF 3644]

Denote by $\pi(x)$, $p(n)$ and $\theta(x)$, respectively, the number of primes not exceeding x, the nth prime and the logarithm of the product of all primes not exceeding x. It is known that for each positive constant A there exists a constant N for which the following statements are true:

$$\frac{x}{\log x - 1 + A} < \pi(x) < \frac{x}{\log x - 1 - A}, \qquad x \geqq N,$$

$$n\log n + n\log\log n - n - An < p(n) < n\log n + n\log\log n - n + An, \qquad n \geqq N,$$

$$\left(1 - \frac{A}{\log x}\right)x < \theta(x) < \left(1 + \frac{A}{\log x}\right)x, \qquad x \geqq N.$$

The author considers two problems: (i) that of determining the N which goes with a particular A; (ii) that of determining how small A can be without requiring that N be large. Partial answers are given of which the following are sample results:

$$(1)\quad \frac{x}{\log x} < \pi(x) < \frac{x}{\log x - 2}, \qquad 17 \leqq x \leqq e^{100};\ x \geqq e^{2000},$$

$$(2)\quad \frac{x}{\log x + 2} < \pi(x) < \frac{x}{\log x - 4}, \qquad x \geqq 55.$$

It is to be noticed from (1) and (2) that there is a gap from e^{100} to e^{2000} for which we must be satisfied with $A=3$ instead of $A=1$. The author explains that this situation is apparently due to the insufficiency of our information about the zeros of the Riemann zeta function. For $x \leqq 10^6$ the proof is based on comparisons with Lehmer's "List of Prime Num-

bers" and the ingenious use of a computing machine. For $x \geqq 10^6$ analytical methods are used. These are based on methods discussed in Landau's "Handbuch der Lehre von der Verteilung der Primzahlen," with the later improvements of Backlund, Ingham, Titchmarsh and the author.

R. D. James (Saskatoon, Sask.).

Petersson, Hans. Konstruktion der sämtlichen Lösungen einer Riemannschen Funktionalgleichung durch Dirichlet-Reihen mit Eulerscher Produktentwicklung. III. Math. Ann. **117**, 277–300 (1940). [MF 2156]

In Part I of this series [Math. Ann. **116**, 401–412 (1939)] the author has, by means of a metrization in the space of the entire modular forms, proved that the matrices $\lambda(n)$ of Hecke's theory of Euler products [Math. Ann. **114**, 1–28, 316–351 (1937)] can, for "Stufe" 1, simultaneously be transformed into diagonal form, which means that Dirichlet series and Euler products with numerical coefficients instead of matrix coefficients can be obtained. The situation for Stufe Q has been studied in Part II [Math. Ann. **117**, 39–64 (1939); these Rev. **1**, 294]. The simultaneous reduction on diagonal form is also possible for the sets of modular forms belonging to a divisor t of Q which contains only prime factors of $t_1 = Q/t$. If, however, t contains primes not dividing t_1, some eigenvalues $\omega(n)$ can be multiple for all operators T_n. In the Euler product of a Dirichlet series which corresponds to such a modular form they give rise to a factor $K(s)$, here called the "kernel." The result of this paper is that the kernel is always a linear combination of a finite number of products of the form

$$q_1^{-(\nu_1-1)s}\left(1-\frac{\alpha_1}{q_1^s}\right)^{-\nu_1}\cdots q_k^{-(\nu_k-1)s}\left(1-\frac{\alpha_k}{q_k^s}\right)^{-\nu_k},$$

where $q_1, \cdots, q_k$ are the primes dividing t but not t_1, $\alpha_1, \cdots, \alpha_k$ characteristic roots of certain linear transformations and $\nu_1, \cdots, \nu_k$ integers which can run up to the multiplicity of the corresponding α at most. For $k=1$ the author proves moreover that each

$$q^{-(\nu-1)s}\left(1-\frac{\alpha}{q^s}\right)^{-\nu}$$

is itself an admissible $K(s)$; this case takes place in particular for $Q=t=q^n$. *H. Rademacher* (Swarthmore, Pa.).

Scott, S. A. Some applications of the generalised Poisson-Jensen formula. Proc. Edinburgh Math. Soc. (2) **6**, 151–156 (1940). [MF 2847]

The formula in question is

$$-2\pi\sum_\mu g(a_\mu)+2\pi\sum_\nu g(b_\nu)=\iint_G \log|f(s)|\,\Delta g(s)\,d\Sigma + \int_\Gamma\left\{\log|f(s)|\frac{\partial g(s)}{\partial n}-g(s)\frac{\partial \log|f(s)|}{\partial n}\right\}du,$$

where $f(s)$ is meromorphic with zeros a_μ and poles b_ν in the domain G with boundary Γ; $g(s)=g(\sigma+ti)$ is a real function of σ and t with continuous partial derivatives up to the second order, and $\partial/\partial n$ is taken along the inside normal of Γ. In all applications the function $f(s)$ is taken as $\zeta(s)$, and by different specializations of $g(s)$ the author obtains formulae containing the non-trivial zeros of $\zeta(s)$, as, for example, the Riemann-von Mangoldt formula for $N(T)$. The following formula seems to be new:

$$(*)\qquad \sum_{\beta_\mu \geqq \frac{1}{2}} \log|\chi(\rho_\mu)| = -2\int_2^T \log|\zeta(\tfrac{1}{2}+ti)|\,\Re\left\{\frac{\zeta'}{\zeta}(\tfrac{1}{2}+ti)\right\}dt + O(\log^2 T),$$

where the sum is taken over all zeros $\rho_\mu=\beta_\mu+i\gamma_\mu$ of $\zeta(s)$ with $2<\gamma_\mu<T$, $\chi(s)$ being

$$\zeta(1-s)/\zeta(s)=2(2\pi)^{-s}\cos\frac{\pi s}{2}\Gamma(s).$$

The left-hand member of (*) can be replaced by

$$\sum_{\beta_\mu \geqq \frac{1}{2}} (\beta_\mu-\tfrac{1}{2})\log(\gamma_\mu/2\pi).$$

H. Rademacher (Philadelphia, Pa.).

GEOMETRY

Loong, Chi-Ho. Further generalizations of Simson line, Kantor point and Kantor line. Tôhoku Math. J. **46**, 173–180 (1939). [MF 1181]

Minoda, Takashi. On some theorems concerning S. Kantor's theorem and its extension. Tôhoku Math. J. **46**, 26–40 (1939). [MF 1164]

Lob, H. A note on Morley's trisector theorem. Proc. Cambridge Philos. Soc. **36**, 401–413 (1940). [MF 2919]

The author has previously shown [Proc. Cambridge Philos. Soc. **29**, 45 (1933)] that the chain of circles theorem of De Longchamps, also discovered by F. Morley, may be derived by successive projection of a rational normal C_n in an $[n]$. In the present paper he shows that the consideration of the osculating developables of the various space curves formed by the successive projection leads to properties which in the case of $C_n=C_3$ furnish a proof of Morley's trisector theorem. The paper concludes by treating the latter case in detail. [Cf. also Lebesgue, Enseignement Math. **38**, 39–58 (1940); these Rev. **2**, 8.]

N. A. Court (Norman, Okla.).

Egerváry, Eugen. Über ein räumliches Analogon des Sehnenvierecks. J. Reine Angew. Math. **182**, 122–128 (1940). [MF 2820]

The author extends to space Ptolemy's theorem, in the following way: If a hexahedron with concurrent diagonals is inscribed in a sphere, the sum of the squares of the geometric means of the tetrads of homologous edges is equal to the geometrical mean of the diagonals of the solid.

N. A. Court (Norman, Okla.).

van Wijk, U. H. The problem of contact of Apollonius. Mathematica, Zutphen. A. **9**, 157–159 (1940). (Dutch) [MF 3706]

Montel, Paul. On the quadrature of the circle. An. Soc. Ci. Argentina **128**, 321–330 (1939). (Spanish) [MF 3771]

Daus, P. H. Bisecting circles. Amer. Math. Monthly **47**, 519–529 (1940). [MF 3076]

The families of circles are considered which (a) bisect two circles, (b) are bisected by two circles, and (c) bisect

one circle and are orthogonal to another. The results are applied to ruler and compass construction of circles, when bisection is one of the determining conditions.

N. A. Court (Norman, Okla.).

Tietze, Heinrich. Über die mit Lineal und Zirkel und die mit dem rechten Zeichenwinkel lösbaren Konstruktionsaufgaben. I. Math. Z. **46**, 190–203 (1940). [MF 2395]

Given $n+1$ points $A_0, A_1, \cdots, A_n$ not all collinear and with Cartesian coördinates $(0, 0), (a_1, b_1), \cdots, (a_n, b_n)$, respectively, constructions with a "square" (instrument for drawing straight lines and right angles) generate the points whose coordinates are: $x=\sum_{s=1}^{n} k_s a_s$, $y=\sum_{s=1}^{n} k_s b_s$; and the lines whose equations are: $ux+vy+w=0$, with u, v of the same form as the above x, y, respectively, and with $w=k_0 r_{11}$, where k_0, k_s belong to the field generated by 1, r_{12}/r_{11}, $r_{22}/r_{11}, \cdots, r_{nn}/r_{11}$ with $r_{st}=a_s a_t+b_s b_t$. *J. L. Dorroh.*

Garnier, René. La formule de Savary et la construction de Bobillier en géométrie plane hyperbolique. Bull. Sci. Math. **63**, 279–300 (1939). [MF 973]

Terheggen, H. Zur analytischen Geometrie auf der Geraden von Hermite als Grenzfall der Geometrie in der Hermitischen Ebene und ihr Zusammenhang mit der gewöhnlichen sphärischen Trigonometrie. Jber. Deutsch. Math. Verein. **50**, 24–35 (1940). [MF 2771]

The author developed in an earlier paper [W. Blaschke and H. Terheggen, Rend. Sem. Mat. Roma **3**, 153–161 (1939); cf. these Rev. **1**, 261] the geometry of a triangle in a Hermitian plane. If the three points are on a "normal chain," the ordinary spherical trigonometry is obtained. In this paper the author investigates the special case that the three points are on a straight line of the Hermitian plane, which case cannot be settled by the methods of the earlier paper. The points of the Hermitian line can be mapped on the points of a real sphere, and thus a new connection between the Hermitian geometry and the spherical trigonometry is developed. Not only the elementary formulas but the entire spherical trigonometry as developed by Study can be obtained in this way. *E. Helly.*

Ficken, F. A. Cones and vector spaces. Amer. Math. Monthly **47**, 530–533 (1940). [MF 3077]

The author gives a lattice-theoretic interpretation of various concepts of abstract projective geometry. Included are the concepts of polarity with respect to a non-singular quadratic form, and related positiveness, negativeness, and isotropy of vectors, and decompositions into subspaces. Proofs are omitted because they follow well-known patterns.

G. Birkhoff (Cambridge, Mass.).

Haenzel, G. Geometrie und Wellenmechanik. II. Diracsche Gleichung, Unschärfe und Vertauschbarkeit. Jber. Deutsch. Math. Verein. **50**, 121–129 (1940). [MF 3072]

In a previous paper [in the same Jber. **49**, 215–242 (1940); these Rev. **1**, 262] the author explained how Eddington's operators E_{ik} may be associated with the Pascal configuration. In the first part of the present paper he dualizes this result, and shows how these operators may be associated with the Brianchon configuration. In the remainder of the paper an attempt is made to interpret the "uncertainty principle" geometrically. *G. de B. Robinson.*

Ramamurti, B. A geometrical proof of a theorem on spinors. Proc. Indian Acad. Sci., Sect. A. **11**, 497–501 (1940). [MF 2688]

The author obtains the known relation between symmetric second-order two-component spinors and self-dual antisymmetric four-dimensional two-index tensors by using Veblen's geometric representation [Rice Inst. Pamphlet (1934)] of two-component spinors as homogeneous parameters for the two families of generators of the quadric which is the section of the null cone $(x^1)^2+(x^2)^2+(x^3)^2-(x^4)^2=0$ by the hyperplane at infinity. *A. H. Taub.*

Bottema, O. Self-projective point-sets. Nederl. Akad. Wetensch., Proc. **43**, 591–598 (1940). [MF 3101]

If we consider, in n-space S_n, a set of $n+2$ points, no $n+1$ of which belong to a S_{n-1}, taken in a given order, there always exists a non-singular collineation which interchanges the points of the set in a given way. This theorem does not hold for a set of $n+3$ points taken in a given order and which may be called a throw. If the case $n=1$ is excluded, there does not generally exist a collineation different from identity so that the set, taken as a whole, is not altered. The question arises of constructing sets which are invariant for certain finite collineation groups. For $n=2$ and $n=3$, the problem was completely solved by Barrau [Nederl. Akad. Wetensch., Proc. **39**, 955–961 (1936); **40**, 150–155 (1937)]. The author gives a solution for the general case. The method is based on the fact that $n+3$ points of a throw in S_n always lie on a rational normal curve C_n of degree n, and correspond to $n+3$ values of a complex parameter t. If the given throw is self-projective, then there exists a group of linear transformations of the variable t which leaves the corresponding points of the complex plane invariant. The theory of all finite subgroups of the group of linear transformations of a complex variable is known [cf. F. Klein, Vorlesungen über das Ikosaeder, Leipzig, 1884]. Using these known results, the author gives a method for constructing all self-projective throws in S_n. He enumerates all possible cases for S_2, S_3 and S_4. For general n, each case must be considered for itself, the solution depending on the arithmetic properties of the number n. *E. Helly.*

Bottema, O. Ueber affine Invarianten bei quadratischen Formen. Nederl. Akad. Wetensch., Proc. **43**, 866–873 (1940). [MF 3181]

Weitzenböck has investigated the invariants of a quadratic and a linear form, in 2-space and 3-space, when the transformations of the affine group are applied. The author replaces Weitzenböck's absolute invariants by other functions, which have certain advantages. He discusses the geometrical meaning of his absolute invariants in 2-space, and shows how his results can be transferred to the system of a quadratic and a linear form in n-space. *E. Helly.*

Bottema, O. Absolute Invarianten von fünf Geraden in R_4. Nederl. Akad. Wetensch., Proc. **43**, 1011–1015 (1940). [MF 3531]

Absolute invariants of five straight lines l_1, l_2, l_3, l_4, l_5 in 4-space can be found by computing the ratios of certain ordinary invariants of the five lines as given by Weitzenböck [Nederl. Akad. Wetensch., Proc. **42**, 245–252 (1939)]. The author of the present paper shows that these absolute invariants are equal to certain crossratios connected with the given lines in that way: the lines l_1, l_2, l_3 have generally one transversal line m_{45}; the lines m_{45}, l_4, l_5 have a transversal line

n_{45}. The line m_{45} is met by the lines l_1, l_2, l_3 and n_{45} in four points, whose crossratio is an absolute invariant of the five given lines. There are, in general, 10 such crossratios, which are determined by five of them. *E. Helly.*

van der Woude, W. Ueber vier Gerade in R_4. Nederl. Akad. Wetensch., Proc. **43**, 946–954 (1940). [MF 3529]

A well-known theorem [cf. H. F. Baker, Principles of Geometry IV, Cambridge] states that, if four general straight lines a_1, a_2, a_3, a_4 in four-space are given, one and only one associated line a_5 can be found so that (condition (A)) each plane intersecting the four given lines also intersects the fifth line a_5. The expression "four general lines" means that (a) no three of these lines are in the same 3-space, (b) there is no line intersecting all four lines, (c) there is no plane through one of the four lines which intersects the other three lines. The author investigates the consequences resulting if one of these conditions is not satisfied. (a) If a_1, a_2, a_3 are in the same 3-space, all lines of the regulus which is determined by a_1, a_2, a_3 satisfy condition (A). (b) If there is a line intersecting a_1, a_2, a_3, a_4, all lines of a certain ruled surface of the third order satisfy (A). (c) If there is a plane through a_4 intersecting a_1, a_2, a_3, the line a_5 coincides with a_4. *E. Helly* (Paterson, N. J.).

Graf, H. Räumliche Ebenennetze, deren Sechsflachzellen Inkugeln besitzen. Math. Z. **46**, 591–604 (1940). [MF 2784]

Generalizing his own article in the plane, written in collaboration with R. Sauer [Math. Z. **44**, 374–378, 380–386 (1938)], the author proves the proposition: Ein ebenflächiges Sechsflachnetz, dessen sämtliche Einzelzellen Inkugeln besitzen, hat stets die Eigenschaft, dass auch sämtliche n^3-Zellen beliebiger Grösse ($n=2, 3, 4, \cdots$) Inkugeln haben. *N. A. Court* (Norman, Okla.).

Wells, A. F. Finite complexes in crystals: a classification and review. Philos. Mag. (7) **30**, 103–134 (1940). [MF 2766]

By tabulating non-negative integral solutions of the equations $N_2+N_3+N_4=p$, $2N_2+3N_3+4N_4=2l$, for integral values of p from 3 to 12 and of l from p to $2p$, the author provides an arithmetical basis for the enumeration of closed networks or graphs "p_l," having p vertices of which N_2 belong each to two of the l edges, N_3 to three, N_4 to four (and, as a practical limitation, none to more than four). He restricts consideration to planar graphs (thus ruling out 5_{10} and 7_{14}) and then to polyhedra ($N_2=0$). His list of 23 polyhedra with three edges at each vertex, from 4_6 (the tetrahedron) to 12_{18}, agrees with M. Brückner's [Vielecke und Vielflache, 1900, Tafel II]; but his list of 15 polyhedra with four edges at each vertex, from 6_{12} (the octahedron) to 12_{24}, is incomplete, since in fact there are more than two polyhedra 12_{24} with eight triangular and six quadrangular faces. The paper is illustrated with excellent drawings.

H. S. M. Coxeter (Toronto, Ont.).

Brooks, R. L., Smith, C. A. B., Stone, A. H. and Tutte, W. T. The dissection of rectangles into squares. Duke Math. J. **7**, 312–340 (1940). [MF 3404]

A squaring of order n (n finite) of a rectangle R is a decomposition of R into n non-overlapping squares. If $n>1$, and if the squares are all unequal, the squaring is called perfect, and R is a perfect rectangle. Dehn and Sprague have proved that a rectangle can be squared if and only if its sides are commensurable; it then is perfectible in an infinity of essentially different ways [Dehn, Math. Ann. **57**, 314–332 (1903); Sprague, J. Reine Angew. Math. **182**, 60–64 (1940); Math. Z. **46**, 460–471 (1940); cf. these Rev. **2**, 11]. The authors associate with each "oriented" squared rectangle a certain linear graph and a flow of electric current through it and study the squared rectangles by means of the associated electric networks. They not only obtain new proofs of the theorems mentioned above but many other results such as methods of construction of (perfect) squared rectangles and squares, and sufficient conditions for a squaring to be perfect. They prove $\min n=9$ for all perfect rectangles, and $\min n\leqq 26$ for perfect squares. The last paragraph contains some generalizations. *P. Scherk* (New Haven, Conn.).

Perron, Oskar. Über lückenlose Ausfüllung des n-dimensionalen Raumes durch kongruente Würfel. Math. Z. **46**, 1–26 (1940). [MF 1474]

Bei seiner Untersuchung der Minkowskischen Vermutung über den Grenzfall des Minkowskischen Satzes über homogene Linearformen, hat O. H. Keller diese Vermutung verallgemeinert zu einer Aussage über das Auftreten gewisser Würfelpaare bei lückenloser Ausfüllung des R_n durch kongruente Würfel [J. Reine Angew. Math. **163**, 231–248 (1930)]. Die Kellersche Vermutung wird vom Verfasser eingehend untersucht. Gezeigt wird, dass für jedes feste n die Entscheidung über Richtigkeit oder Falschheit der Vermutung durch endlich viele Versuche erzwungen werden kann. *J. F. Koksma* (Amsterdam).

Perron, Oskar. Modulartige lückenlose Ausfüllung des R_n mit kongruenten Würfeln. I. Math. Ann. **117**, 415–447 (1940). [MF 3004]

A cube with center $(a^1, \cdots, a^n)$ in Euclidean n-space R_n is the set of points $(x^1, \cdots, x^n)$ with $-\frac{1}{2}\leqq x^\nu-a^\nu<\frac{1}{2}$ ($\nu=1, \cdots, n$). An M-covering of R_n is a set of cubes such that each point of R_n lies in one and only one of them, and such that the vectors from the origin to the centers of the cubes form a modul. Minkowski conjectured that every M-covering contains n cubes with the respective centers $(a_i^1, \cdots, a_i^n)$ such that for conveniently numbered coordinates $a_i^i=1$ and $a_i^k=0$ if $i>k$ ($i, k=1, 2, \cdots, n$). According to a theorem of the author every M-covering contains n cubes with the respective centers $(\alpha_i^1, \cdots, \alpha_i^n)$ such that $\alpha_i^i=1$, $0\leqq\alpha_i^k<1$ for $i\neq k$ ($i, k=1, \cdots, n$) [Math. Z. **46**, 1–26 (1940), theorem 5; cf. the preceding review]. In this paper he shows that Minkowski's hypothesis is equivalent to the conjecture that all the products

$$\alpha_{\lambda_1}^{\lambda_2}\alpha_{\lambda_2}^{\lambda_3}\cdots\alpha_{\lambda_{r-1}}^{\lambda_r}\alpha_{\lambda_r}^{\lambda_1}$$

vanish ($r=2, \cdots, n$; $\lambda_i\neq\lambda_k$ for $i\neq k$; $i, k=1, \cdots, r$). He proves that conjecture for $n\leqq 8$. His main tool is his theorem that, if the center of a cube of an M-covering is not the origin, at least one of its coordinates is an integer not equal to 0. *P. Scherk* (New Haven, Conn.).

Strubecker, Karl. Beitrag zum Pohlkesatz. Akad. Wiss. Wien, S.-B. IIa **148**, 107–110 (1939). [MF 2538]

Pohlke's well-known theorem states: let $\Delta'=(U', A', B', C')$ be a system of four non-collinear points on a plane. Then a "tripod" $\Delta=(U, A, B, C)$ with $UA=UB=UC$, $UA\perp UB\perp UC$ can be found such that Δ' is an oblique projection of Δ. There are many proofs of this theorem. The new proof given in this paper, based on a paper of E. Kruppa

[Akad. Wiss. Wien, S.-B. 116, 931–936 (1907)], uses freely the description of metric properties of figures in terms of their relations to the absolute polar-system. *E. Helly.*

Graf, Ulrich. Die Projektion bewegter Bilder auf gewölbte Schirmflächen. Z. Angew. Math. Mech. **20**, 50–57 (1940). [MF 2375]

Graf, Ulrich. Pathologische Perspektiven. Jber. Deutsch. Math. Verein. **50**, Abt. 2., 35–53 (1940). [MF 3075]

Experience shows that exactly constructed perspective drawings give a kind of unnatural impression, in increasing degree the larger the distance from the fundamental point (pedal point of the normal line of vision) becomes. The ideal impression would appear on a sphere around the eye ("Sehkugel"). In order to obtain an image which corresponds more to our visual impression, two successive mappings are considered: the first of the object on the "Sehkugel," the second of the "Sehkugel" on the plane. If this second mapping is different from the perspective mapping of the "Sehkugel" from its center on a tangent plane the resulting image is called a "pathological perspective picture." Four different kinds of such pathological mappings are discussed.

O. Neugebauer (Providence, R. I.).

Gotthardt, E. Beiträge zur Frage der Genauigkeit der gegenseitigen Ortung von Senkrechtbildpaaren. Bildmessung und Luftbildwesen. Beilage Allg. Vermessgs.-Nachr. **15**, 2–24 (1940). [MF 2763]

The errors of correspondence setting of two aerial photographs can be found by the method of least squares using only the ordinates of at least five couples of corresponding points. The errors are defined as small rotations around the three axes of swing, setting and tilt. The least square method gives the most probable values of these quantities and their probable errors. The problem of this paper is to investigate whether these rotations can be replaced by certain linear functions of them, whose probable errors have extreme values. Let $X_1, X_2, \cdots, X_s$ be the variables and $F=\sum f_n X_n$ a linear function, normed so that $\sum f_n^2=c^2$. The precision of this function is given by a quadratic form $Q=\sum Q_{mn} f_m f_n$, and this value has to be a maximum or minimum while $\sum f_n^2=c^2$. This problem is identical with the well-known transformation of a quadratic form into a sum of squares, and, generally, five functions F_m can be found satisfying the given conditions. These functions can be introduced, as a kind of normal coordinates, to replace the quantities X_n. The author investigates the significance of these functions with respect to the mapping problem, using a numerical example, and he shows the advantage of introducing these normal-coordinates in order to study the deformation of the model due to errors of orientation. *E. Helly.*

Frank, M. L. Descriptive geometry of a four-dimensional space according to the concept of E. S. Fedoroff. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **6**, 90–107 (1939). (Russian) [MF 3293]

Differential Geometry

***Eisenhart, Luther Pfahler. An Introduction to Differential Geometry.** Princeton Mathematical Series, v. 3. Princeton University Press, Princeton, N. J., 1940. x+304 pp. $3.50.

The book deals with classical differential geometry of curves and surfaces in Euclidean 3-space. The first chapter is mainly concerned with curvature, torsion, the Frenet formulas for a space curve and some surfaces associated with a curve. The second chapter is devoted to tensor calculus and also contains a rather complete discussion of the system of total equations $d\theta^a=\psi_i^a(\theta, x)dx^i$ on which some theorems in the sequel are based. The third chapter is devoted to the first fundamental form and the intrinsic geometry of surfaces, while the fourth and last chapter treats the second fundamental form, the total and mean curvatures and the equations of Gauss and of Codazzi and Mainardi. Many topics are treated all too briefly and it is probably due to this that a number of inaccuracies have crept in.

M. S. Knebelman (Pullman, Wash.).

v. Sz. Nagy, Gyula. Über die Eigenschaften der beschränkten ebenen Kurven ohne Tangentensingularität. Math. Z. **46**, 605–626 (1940). [MF 2785]

Given a curve in the Euclidean plane with continuous tangents everywhere; the order [class] of that curve is defined as the maximum number of its points [tangents] which coincide with a straight line [point]. A curve K_r shall be closed and shall have no other singularities than double points and r cusp points of the first kind. There are two types of curves K_r according as r has even and odd values. The curves K_r for which r is even are H. Brunn's "Buschenveloppen" B_r. They possess exactly one tangent in each direction [Brunn, Ueber Kurven ohne Wendepunkte, München, 1889; v. Sz. Nagy, Math. Z. **41**, 479–492 (1936)]. If r is odd, we obtain curves C_r which have exactly two tangents in each direction. The r cusp points of a curve C_r decompose it into r components; $r-1$ of them are convex; the rth component is either convex or of third order and may be decomposed then into two convex arcs. A line of a pencil of parallel straight lines meets either one of the components in no more than three points or two of them in no more than two points each, all the others in both cases in at most one point each. Hence the order of a curve C_r is not greater than $r+2$; it is at least equal to its class. Furthermore, the author discusses the position and number of the double points and some other properties of a curve C_r, as well as various constructions of and at curves K_r such as small alterations in the neighborhood of a cusp point or double point, through which a curve B_{r+1} is transformed into a curve C_r or a curve C_r is dissected into two curves, one of which is a curve B_{r_1}, while the other is a curve C_{r_2} $[r_1+r_2=r+1]$ if it has no double tangents.

P. Scherk (New Haven, Conn.).

Markoff, A. A. What is a smooth surface? Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **10**, 27–39 (1940). (Russian) [MF 3301]

Defining differential geometry as the study of geometric objects by the methods of infinitesimal analysis, the author seeks to delimit in a purely geometric way the class of objects which underlie this subject. He defines a smooth manifold X in an affine r-dimensional space A_r as a connected and locally closed subset of that space which has at each point $a \in X$ a "tangent space" $T(X, a)$. This, in turn, is defined essentially as follows. For a given set X in A_r, $x \in X$, and positive real ξ, let $H(X, x, \xi)$ denote the set which results from the "magnification" of X in the ratio ξ with the point x as center: that is, the locus of endpoints of vectors ξv, where v is a radius vector with origin at x and terminus on X. Now, a point y of A_r is said to be an upper limiting point of the system $H(X, x, \xi)$ as $x \to a$ and $\xi \to \infty$,

provided that, for an arbitrary choice of a neighborhood U of y, neighborhood V of a, and real τ, there exists at least one point $x \varepsilon V \cap X$ and one real $\xi > \tau$ such that (*) $U \cap H(X, x, \xi) \neq 0$. The point y is said to be a lower limiting point of that same system provided that to the given U of y there exists a V of a and a τ such that (*) holds for every $x \varepsilon V \cap X$ and every $\xi > \tau$. When it happens for a given set X and point $a \varepsilon X$ that every upper limiting point of the system $H(X, x, \xi)$, $x \to a$, $\xi \to \infty$, is also a lower limiting point, then the totality of these limiting points is called the limit set. This limit set is, by definition, the tangent space $T(X, a)$. The tangent space, when it exists, is a linear manifold (in A_r) containing the point a (Th. 1). If X is a smooth manifold, $T(X, a)$ varies continuously with a (Th. 2) and is of constant dimension; in anticipation, this is called the dimension of X. The cardinal objective of the paper is attained in the theorem (5) that at each point of an n-dimensional smooth manifold X in A_r there exists a neighborhood U such that in the associated coordinate system the set $U \cap X$ is described by the equations:

$$\xi_i = f_i(\xi_1, \xi_2, \cdots, \xi_r), \qquad i = n+1, \cdots, r,$$

where the functions f_i are defined in an open subset of the r-dimensional number space, and possess continuous first partial derivatives with respect to all arguments. The principal tool in the construction of these functions is the use of families of parallel linear manifolds of dimension $r-n$ transversal to the tangent space. *L. Zippin.*

Alexandroff, A. D. Almost everywhere existence of the second differential of a convex function and some properties of convex surfaces connected with it. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. 6, 3–35 (1939). (Russian) [MF 3285]

The following results, which for two variables were obtained by H. Busemann and W. Feller [Acta Math. 66, 1–47 (1936) and Bemerkungen zur Differentialgeometrie der konvexen Flächen. III., Mat. Tidsskr. B. 1936, 41–70], are established for n variables: A convex function $f(x_1, \cdots, x_n) = f(x)$ has almost everywhere a generalized second differential. This means geometrically that at almost every point $(x^0, f(x^0))$ of the convex surface $\Sigma : z = f(x)$ the following statements hold: The tangent plane π exists in the strong sense that it is the paratingent. Every two-dimensional plane through $(x^0, f(x^0))$ which is not contained in π intersects the surface in a curve which has a finite curvature at $(x^0, f(x^0))$. The curvatures obey the laws of Meusnier and Euler. The limit process by which the curvature is defined is uniform in a certain sense. If σ is a set on Σ we call the spherical image of σ the set $\omega(\sigma)$ of all points of $\sum x_i^2 + z^2 = 1$ where the normal is parallel to a normal of Σ at a point of σ. If this mapping of Σ into $z^2 + \sum x_i^2 = 1$ is absolutely continuous the measure of $\omega(\sigma)$ can be expressed as the integral over the product of the main curvatures (which are almost everywhere defined). Furthermore the following result on "mixed volumes" (gemischte Volumen) is obtained: If $H_1, \cdots, H_n$ are the supporting functions of convex bodies, if the functions $\sum \Lambda_i H_i$ are absolutely continuous for $\Lambda_i > 0$ and if the radii of curvature of the corresponding surfaces never vanish, then

$$V(H_1, \cdots, H_n)^2 \geqq V(H_1, \cdots, H_{n-1}, H_{n-1}) V(H_1, \cdots, H_{n-2}, H_n, H_n),$$

and the equality sign holds only if the bodies are homothetic. *H. Busemann* (Chicago, Ill.).

Balanzat, Manuel. Généralisation de quelques formules de géométrie intégrale. C. R. Acad. Sci. Paris 210, 596–598 (1940). [MF 3033]

Balanzat, Manuel. Integral formulas for the intersection of sets. Union Mat. Argentina, Publ. no. 14, 19 pp. (1940). (Spanish) [MF 2940]

Given two measurable sets E_1 and E_2 with respective measures $m(E_1)$ and $m(E_2)$ in an Euclidean n-space ($n = 1, 2, 3$). Let dK be the kinematic density in that space. Let E_1 be movable, E_2 fixed, and let $m(E_1, E_2)$ be the measure of their crosscut. Then

$$\int m(E_1, E_2) dK = C m(E_1) m(E_2),$$

where C is equal to 1, 2π or $8\pi^2$ according as n is 1, 2 or 3. This formula holds if the sets are intervals, squares and cubes [Blaschke, Integralgeometrie, I and II], and therefore also for open sets; finally either of the sets E_1 and E_2 may be represented as the difference of an open set and a set of arbitrarily small measure. Analogous formulas hold if one of the sets consists of a finite number of points, or if it is replaced by a straight line, a plane, a plane band between two straight lines, a band between two parallel planes, a cylinder, etc., or if E_1 is subjected only to translations. *P. Scherk* (New Haven, Conn.).

Kasner, Edward and De Cicco, John. General invariants of irregular analytic elements. Proc. Nat. Acad. Sci. U. S. A. 27, 88–92 (1941). [MF 3660]

An analytic element is an analytic arc associated with one of its points, and is represented by expressing the coördinates as integral power series in a parameter. This study classifies analytic elements with respect to the group of regular point transformations of the plane. Absolute differential invariants are found for all elements except those of a limited number of types. These types are specified, and criteria of equivalence are given for them. *P. Franklin.*

Kasner, Edward and De Cicco, John. The conformal near-Moebius transformations. Bull. Amer. Math. Soc. 46, 784–793 (1940). [MF 2923]

By near-Moebius transformations the authors mean those conformal transformations which take $2\ \infty^1$ circles into circles. This is motivated by the fact that, if $3\ \infty^1$ circles go into circles, the conformal transformation is a Moebius transformation. The authors determine canonical forms for all near-Moebius transformations. These forms show that all near-Moebius transformations are simple combinations of Moebius transformations and one of the three transformations e^z, $\log z$ or z^n, and that the two families of circles which go into circles are orthogonal pencils.

P. Franklin (Cambridge, Mass.).

Narasinga Rao, A. Studies in circle geometry. Math. Student 8, 53–72 (1940). [MF 3281]

The first few paragraphs of this paper contain a survey of the geometries of the circle connected with the names of Moebius, Laguerre and Lie. The author then reports the results of ten papers which he has published since 1936. They deal partly with generalizations of certain systems of circles, called Miquel-Clifford configurations, and partly with Kasner's "turbine" geometry (a turbine being defined as an aggregate of oriented surface elements, whose point-locus is a sphere C, and whose plane-locus is a concentric sphere c). In the last paper the author studied the topology

of the totality of line-elements in the inversive plane. The author indicates, in the concluding chapter, directions in which the investigations may be carried further.

E. Helly (Paterson, N. J.).

Maeda, Jusaku. Geometrical meanings of the inversion curvature of a plane curve. Jap. J. Math. **16**, 177–232 (1940). [MF 2598]

The author determines various sets of circles as loci connected with sets of loxodromics having at least four- or five-point contact with a curve and interprets the inversion length and inversion curvature geometrically. The following example seems the most interesting one: An isogonal trajectory of the circles through two given points a and b is called a loxodromic with the poles a and b. Let a and b be the poles of any loxodromic having at least five-point contact with a given curve at a point p of the curve. The circle through a, b and p and the circle through p with respect to which a and b are mutually inverse determine a coaxial system of circles. The circle of this system which is orthogonal to the curve in p is called the normal circle of the curve in p. It does not depend upon the choice of the loxodromic. If $d\theta$ is the angle between the normal circles of the curve at the points p and $p+dp$ (dp=inversion length), then the inversion curvature at p equals $(d\theta/dp)^2$. The results are obtained by elementary computations.

P. Scherk (New Haven, Conn.).

Maeda, Jusaku. On the Laguerre-geometry of plane curves. Jap. J. Math. **17**, 13–25 (1940). [MF 3153]

The paper derives the well-known representation of Laguerre transformations as linear transformations with dual numbers. Furthermore the notation with dual numbers is used for the exposition of Kubota's method of determination of the fundamental invariants of a plane curve in Laguerre geometry. [Cf. Blaschke's Differentialgeometrie, v. 3, §§ 38, 41.] *P. Scherk* (New Haven, Conn.).

Maeda, Jusaku. On the theory of curves in Euclidean three-space. Sci. Rep. Tôhoku Imp. Univ., Ser. 1. **28**, 319–333 (1940). [MF 1710]

The author computes from the Frenet formulas some loci and the natural equations of various curves with given relations between their moving Euclidean and affine trihedrals, He thus characterizes the curves for which $\tau=\text{const.}$, $\kappa^2\tau^{-1}=\text{const.}$, $\kappa^2\tau=\text{const.}$, $s+b=a(\sigma+c)^2$ (a, b, c=const., s=arc length, κ=curvature, τ=torsion, σ=affine arc-length). He also computes the natural equations of the curves with identically coinciding affine and Euclidean binormals. *P. Scherk* (New Haven, Conn.).

Maeda, Jusaku. On the theory of curves in affine space. Sci. Rep. Tôhoku Imp. Univ., Ser. 1. **28**, 350–369 (1940). [MF 1712]

Let a straight line through a moving point on a curve in affine 3-space be fixed affinely to the principal (or Wintern-itz-) trihedral. The author gives necessary and sufficient conditions that the line describes a developable surface. Furthermore, he determines the loci of lines through a fixed point of the curve along which the curve is projected into a plane curve with given properties. Finally, he discusses some properties of the ruled surfaces described by the affine principal normals (or binormals) of a curve.

P. Scherk (New Haven, Conn.).

Maeda, Jusaku. On some osculating figures of a plane curve, and on sections of a surface by planes passing through a fixed tangent. Sci. Rep. Tôhoku Imp. Univ., Ser. 1. **29**, 163–203 (1940). [MF 3175]

Let the plane curve (M) be generated by a point M, and C be its center of curvature. A point P_n is characterized as follows: Let Z_n be a point on the normal to (M) at M such that $MC:MZ_n=n:1$. Let (Z_n) be the circle with center at Z_n tangent to (M) at M. The envelope of (Z_n) as M moves along (M) is (M) and the locus of the point P_n. The locus of P_n as n varies (M fixed) is a nodal circular cubic, the nodal tangents being the tangent and normal of (M) at M. Curves having various orders of contact with (M) are discussed. Among these are rectangular hyperbolas, equilateral spirals, hypocycloids, epicycloids, cycloids, tractrices and catenaries each having three-point contact. The point P_n and circle (Z_n) are characterized for particular values of n. For example, the locus of the node of the lemniscates of Bernoulli having at least three-point contact is the circle $(Z_{3/2})$; the locus of the center of rectangular hyperbolas having at least three-point contact is (Z_{-1}). The point $P_{1/4}$ is the focus of the osculating parabola; P_{-1} is the center of the osculating equilateral hyperbola; $P_{3/2}$ is the node of the osculating lemniscate of Bernoulli; and $P_{3/4}$ is the cusp of the osculating cardioid. The last section of the paper is devoted to applications to sections of a surface through a fixed non-asymptotic tangent, and loci of the special points and lines related to the osculating curves previously obtained are discussed. A typical theorem is: If a given surface is cut by a plane π through a non-asymptotic tangent t, the locus of the node of the osculating lemniscate of Bernoulli is a circle as π turns around t. Gaussian coordinates and the methods of functions of a complex variable are employed. *V. G. Grove.*

Popa, Ilie. Sugli osculanti di una curva. Boll. Un. Mat. Ital. (2) **2**, 230–233 (1940). [MF 2978]

Si considera la retta intersezione del piano osculatore in un punto P ad una curva sghemba e del piano congiungente tre punti della stessa curva; esaminando poi le posizione limiti di questa retta secondo il modo di tendere dei tre punti a P. *Author's summary.*

Tsuboko, Matsuji. On the tangents of the reciprocal derivative curves in E_4. Proc. Phys.-Math. Soc. Japan (3) **22**, 509–518 (1940). [MF 2695]

If Γ_x is a curve in a four dimensional projective space E_4 and y is a point of the solid osculating Γ_x, Γ_y is the reciprocal derivative curve of Γ_x if the osculating plane of Γ_y passes through x. This paper investigates some properties of Γ_y with reference to the ruled hypersurface generated by the plane through y and the tangent to Γ_x at x.

M. S. Knebelman (Pullman, Wash.).

Lense, J. Determinazione d'una curva nello spazio euclideo complesso di n dimensioni. Boll. Un. Mat. Ital. (2) **2**, 227–230 (1940). [MF 2977]

Nuova semplificata dimostrazione del modo d'individuare una curva in uno spazio euclideo complesso dato da G. Guichard. *Author's summary.*

Pinl, M. Zur dualistischen Theorie isotroper Kurven. Monatsh. Math. Phys. **49**, 261–278 (1940). [MF 3091]

A new theory of the isotropic curves of a Euclidean complex space R_{2n} whose dimension number is an even integer

$2n$ is developed. In particular, a detailed investigation of the isotropic curves in R_4 is made and the equations of all such isotropic curves is given explicitly. These equations depend upon two arbitrary analytic functions of a complex variable and their first derivatives but do not involve any integrals. Various applications of these equations follow. For example, it is shown that an isotropic curve which is in an isotropic R_3 of R_4 must be in a completely isotropic plane of this R_3. The differential invariants of isotropic curves in R_4 are derived and it is proved that two of these invariants of the third and fourth order respectively constitute a complete basis for all the differential invariants. Various invariant planes and hyperplanes of an isotropic curve of R_4 are defined. A characterization of those isotropic curves one of whose fundamental differential invariants vanishes is stated in terms of these invariant planes and hyperplanes. A number of applications of this theory to minimal surfaces in R_4 are made. *A. Fialkow.*

Blaschke, Wilhelm. Contributi alla geometria projettiva complessa. Boll. Un. Mat. Ital. (2) **2**, 309–314 (1940). [MF 2981]

Projective geometry in the complex field initiated by v. Staudt, Juel and Segre has not progressed in the direction of the corresponding differential geometry. The author proposes some possible lines of such development, limiting himself to the plane. Let X represent a three-vector in the homogeneous complex point coordinates (x), and U a similar vector in line coordinates (u), united position being expressed by $(UX)=0$. Then U and X represent chains of lines and of associated points. If X is a function of two real parameters λ, μ, then $V(X)$ is a two dimensional variety in the plane. Tangency and condition for three-point contact are considered, and expressed in invariant form under projective transformations. The forms are expressed in invariant form, independent of the parameters λ, μ. A few applications of the vector product are made.

V. Snyder (Ithaca, N. Y.).

Terracini, Alejandro. The Mehmke-Segre invariant and linear systems. An. Soc. Ci. Argentina **129**, 97–111 (1940). (Spanish) [MF 3173]

The (projective) invariant of Mehmke-Segre is the ratio of the curvatures of two co-planar tangent curves at the point of tangency [R. Mehmke, Z. Math. Phys. **36**, 206–213 (1891); C. Segre, Atti Accad. Naz. Lincei. Rend. (5) **6**, 168–175 (1897)]. A generalization is the density of a correspondence in the dualistic sense. Given dual coördinates u_i as functions of point coördinates x_k, defining a correspondence, its density is the Jacobian of the u as to the x, divided by the sum of the squares of the nonhomogeneous u to the $(r+1)/2$ power in r-space [Terracini, Atti Accad. Sci. Torino **71**, 310–328 (1936)]. Given a pair of nets of plane curves, the directions at a given point associated with a given value of the Mehmke-Segre invariant define an involution of order 3. The author sets and solves this problem: determine the pairs of nets of conics with a constant Mehmke-Segre invariant. The nets of a pair must have the same Jacobian and the same Cayleyan. Three such pairs (a), (b) exist. I. (a) homaloidal net with three distinct base points, (b) net having these base points as vertices of a self conjugate triangle. II. (a) homaloidal net, conics touching l at L, passing through M, (b) conics touching LM at M, having l for common polar. III. (a) homaloidal net, base points coincident, (b) homaloidal net, conics osculate a different conic. The systems are all self-dual. Besides conics, (b) may include other nets, not all algebraic. Finally, nets of quadric surfaces are considered briefly. *V. Snyder.*

Coolidge, J. L. Analytic systems of central conics in space. Trans. Amer. Math. Soc. **48**, 359–376 (1940). [MF 3161]

In this paper methods of studying systems of conics in space are suggested, and one particular method is given in some detail.

Let the Cartesian coordinates of a point P with respect to a given fixed coordinate system be (X^1, X^2, X^3) and the coordinates of P with respect to a moving system be (x^1, x^2, x^3). Let the coordinates of the origin of the moving system be (X_0^1, X_0^2, X_0^3) referred to the fixed system. The positions of P and the moving axes are assumed to be functions of a parameter v. Then $X^i=X_0^i+a_{ij}x^j$, wherein (a_{ij}) is the orthogonal matrix. The equations of the conic are assumed to be $x^1=a\cos u$, $x^2=b\sin u$, $x^3=0$, wherein a, b are functions of v. In this manner the discussion is made to depend upon the quantities u, v, a, b, $\xi^i=a_{ki}\partial X_0^k/\partial v$, $g_{ij}=a_{ki}\partial a_{kj}/\partial v$.

In section (1) theorems concerning systems of conics tangent to two curves are proved. In particular, a new proof is given of the theorem of Blutel: if the conics of a series of central conics are not coplanar, but touch two curves, the conjugate curves on the surface they generate establish a projective correspondence among the conics. Conditions that the conics of the system be geodesically parallel and that they be lines of curvature are established. In section (2) two parameter systems, or congruences of central conics, are discussed. This is accomplished by considering the conics as determined by two parameters v_1, v_2. A special case of a theorem of Darboux is proved, namely: each conic of a congruence of central conics has normally six focal points. It is also established that if a congruence of central conics is normal to more than six surfaces, the congruence is normal. The semi-axes of a normal congruence of central conics are functionally related. In particular, if the conics of a normal congruence of central conics have axes of fixed length, the locus of their centers is a point, a curve or a surface normal at each point to the plane of the corresponding conic. Conversely, if, in a normal congruence of central conics, the locus of the centers is a surface at each point normal to the plane of the corresponding conic, or is a curve, or is a point, the congruence is generated by conics which are immovable in the moving plane.

V. G. Grove (East Lansing, Mich.).

Vassell, Annette. A complete characterization of sectional families of curves. Amer. J. Math. **62**, 813–822 (1940). [MF 2882]

A sectional family of curves in the plane is obtained by projecting from a fixed point in space upon a fixed plane all the plane sections of an arbitrary surface. Kasner [Bull. Amer. Math. Soc. **14**, 356 (1908); **36**, 51 (1930)] has considered the problem of characterizing such a family of ∞^3 curves. In this paper six properties are found which are necessary and sufficient to characterize a sectional family. The results are applied to developable surfaces and to ruled surfaces. This leads to a new characterization of asymptotic nets on an arbitrary surface and an interpretation of Monge's equation for ruled surfaces. *D. J. Struik.*

Jackson, S. B. The four-vertex theorem for spherical curves. Amer. J. Math. 62, 795–812 (1940). [MF 2881]

This paper contains some analogues on the sphere to developments by Graustein connected with the four-vertex theorem [Graustein, Trans. Amer. Math. Soc. 41, 9–23 (1937)]. *P. Scherk* (New Haven, Conn.).

Horninger, Heinz. Über Fusspunktkurven und -flachen. (Über Fusspunkt- und Gegenpunktflächen II.) Monatsh. Math. Phys. 49, 228–246 (1940). [MF 3089]

This paper studies some kinematical problems in the Euclidean E_3. Let Σ be a motionless and S a movable system. In S there is given a rectangular trihedron with edges e, f, g and vertex P. It is assumed that throughout the movement e contains a given point A. After some general considerations, the author studies the special case where the edge g is the generatrix of a given fixed ruled surface F. A geometrical configuration connected with this movement is defined and its properties are studied. The problem is further specialized to the case that F is algebraic; the pedal curve generated by P is investigated. The author also examines another special case in which P generates a given curve and g belongs to a given complex of straight lines. *G. Fubini* (Princeton, N. J.).

Hamilton, H. J. Notes on curvature of curves and surfaces. Amer. Math. Monthly 47, 613–620 (1940). [MF 3262]

If the plane curve c is sufficiently smooth, P_1 and P_2 points of c on different sides of the point P of c, and A the area bounded by the cord $\overline{P_1P_2}$ and the corresponding arc of c, the number $12A/\overline{P_1P_2}^3$ will tend to the curvature of c at P as P_1 and P_2 approach P. This limit may exist even if the curvature does not exist in the usual sense, and may therefore serve as a generalized curvature. In case the Gauss curvature of a surface is positive at a point P, it can be expressed in a similar fashion. The result obtained in this way follows from two formulae given by Blaschke [see Vorlesungen über Differentialgeometrie I, 2nd ed., pp. 84, 85, formulas (147) and (148)]. *H. Busemann.*

MacQueen, M. L. The axis quadrics at a point of a surface. Amer. J. Math. 63, 112–118 (1941). [MF 3634]

Let there be given on a surface S in S_3 a conjugate net N. The line of intersection of the osculating planes of the curves of the net at a point P of S is called the axis of N at P. The limit of the quadric determined by the axis at P and at two other points on one of the curves of the net through P as these latter points independently approach P along the curve is called by the author an axis quadric. Therefore with the point P and a given conjugate net there is associated a unique pair of axis quadrics. After deriving the equations of these quadrics referred to a convenient coordinate system, several properties of the quadrics are found. Among these properties may be mentioned the following: The quadrics have, of course, the axis at P in common. The projection of the residual cubic curve of intersection from P is a cubic cone with a pair of nodal tangent planes through the axis. These nodal tangent planes intersect the tangent plane to S at P in the tangents to the curves on S corresponding to the developables of the axes congruence. The residual cubic intersects the axis P and also in the focal points of the axis. The tangent plane at the focal point on the tangent at P to one of the curves of N through P to the axis quadric associated with that curve is the plane which corresponds to that focal point in the null-system established by the osculating linear complex of that curve. Finally there is given a geometrical characterization of an harmonic conjugate net. The polar plane of the focal point on the tangent to one of the curves of a conjugate net with respect to the axis quadric associated with the other curve of the net passes through the harmonic conjugate of x with respect to the focal points on the axis if and only if the given conjugate net is harmonic. *V. G. Grove.*

MacQueen, M. L. Conjugate nets and associated quadrics. Duke Math. J. 6, 753–767 (1940). [MF 2729]

The author refers a surface S to a conjugate parametric net N_x. He denotes by C_λ and C_μ the curves through a point P on S of a second conjugate net N_λ distinct from N_x. Quadrics having second order contact with S at P and having third order contact with each or with both of N_λ and N_μ are discussed. There is a pencil Q_λ of these latter quadrics. In particular any quadric of Montard in the direction of a tangent to C_λ at P has third order contact with C_λ at P. This quadric has third order contact with both C_λ and C_μ if and only if N_λ belongs to the Segre-Darboux pencil of conjugate nets. The locus of the polar line of the ray of N_x with respect to the pencil Q_λ is a quadric cone. Using this cone and a covariantly defined cubic curve in the tangent plane at P, certain canonical lines are geometrically characterized. A pair of osculating quadrics of two ruled surfaces associated with the curves of N_x and of C_λ is discussed. The curve C_λ is a union curve of a congruence of lines l if the polar lines of l with respect to these quadrics coincide. *V. G. Grove* (East Lansing, Mich.).

Bachvaloff, S. Quelques remarques sur la méthode du trièdre mobile. Rec. Math. [Mat. Sbornik] N.S. 7 (49), 321–326 (1940). (Russian. French summary) [MF 2794]

Any two-parameter infinitesimal displacement of the moving trihedral is completely determined by certain invariants. By considering a system of completely integrable partial differential equations (a very special system of König) the author finds the generality of these invariants. There is nothing new in this paper. *M. S. Knebelman.*

Roupscheff, I. A. La méthode des variations complexes dans la géométrie sphérique cinématique. C. R. (Doklady) Acad. Sci. URSS (N.S.) 26, 323–326 (1940). [MF 3556]

In this paper the author employs an analytic method to the geometry of a system of geometric figures moving with respect to a fixed point. The method employs two spheres of equal radii, having a common center, the one located with respect to the moving system, the other with respect to a fixed system. By stereographic projection to the points of each sphere there is associated a complex number. To a displacement of the system around the fixed point there is associated a transformation $z=f(\zeta)$ of the complex numbers z, ζ associated with the points of the spheres. *V. G. Grove.*

Schatz, Heinrich. Über Kreisscharen in der Ebene und Kugelscharen im Raum. Monatsh. Math. Phys. 49, 247–260 (1940). [MF 3090]

Verfasser betrachtet ebene Kreisscharen bei denen die Kreismittelpunkte auf einer gegebenen Kurve liegen; dann konstruiert er die beiden Einhüllenden der Kreisschar und die entsprechenden Krümmungskreise K_1, K_2 der Einhüllenden. Der geometrische Ort der Mittelpunkte der Kreise die K_1, K_2 gleichzeitig berühren ist ein Kegelschnitt. Ist dieser Kegelschnitt eine Ellipse oder Hyperbel, so nennt

der Verfasser den Kreis dessen Mittelpunkt im Mittelpunkt des Kegelschnittes und dessen Radius das arithmetische Mittel der Radien der zu Scheiteln des Kegelschnittes gehörenden Kreise ist den "Mittenkreis." Zu drei konsekutiven Kreisen einer ebenen Kreisschar gehören im allgemeinen zwei Krümmungskreise der Einhüllenden. Alle Kreise, welche diese beiden Kreise berühren, bilden eine Ringschar. *T. Kubota* (Sendai).

Schatz, Heinrich. Begleitende Zyklide bei Streifen in der Bewegungsgeometrie. Jber. Deutsch. Math. Verein. **50**, 7–18 (1940). [MF 2769]

In zwei Abhandlungen [Abh. Math. Sem. Hansischen Univ. **5**, 54–84 (1926); Math. Z. **28**, 97–106 (1928); auch W. Blaschke, Vorlesungen über Differentialgeometrie, III, S. 295] hat der Verfasser bewiesen dass bei einem Streifen, der kein Krümmungsstreifen ist, im allgemeinen durch vier konsekutive Flächenelemente eine Zyklide hindurchgeht. Hier ist der Nachweis dieses Satzes in der Bewegungsgeometrie im R_3 geliefert. Am Schluss wird auf den Zusammenhäng mit den oben zitierten zwei Arbeiten eingegangen. *T. Takasu* (Sendai).

Bagchi, Haridas. A note on cyclides. J. Indian Math. Soc. (N.S.) **4**, 120–124 (1940). [MF 3274]

A cyclide is a quartic surface having the circle at infinity as a double curve. It can also be generated as the envelope of a two-parametric family of spheres, whose centers are on a given quartic, called the polar quartic, and which intersect orthogonally a given sphere, called the sphere of inversion. Generally there exist five pairs of focal quadrics and spheres of inversion which determine the same cyclide. The author proves that there is only one focal quadric and one sphere of inversion if the cyclide has a center C, and that C is the common center of the focal quartic and sphere of inversion. Then he investigates loci of points whose polar quadrics with respect to a cyclide have special properties. The author finds also the partial differential equations of the first order for the five systems of ∞^2 spheres which have double contact with the cyclide. *E. Helly.*

Schapiro, J. L. Sur une propriété caractéristique de la métrique des surfaces de rotation. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 23–24 (1940). [MF 3593]

Let there be given on a surface S a net N of curves composed of geodesics. Let $\lambda^i_{1|}$, $\lambda^i_{2|}$ be the unit vector fields determined by the net. Let P^i be the vector of Poisson corresponding to the two vectors $\lambda^i_{s|}$ and let $\eta_i=\lambda_{1|i}+\lambda_{2|i}$. Then if $P^i\eta_i=0$, the given net is such that the distance between any two points on S along the curves of N is independent of the path chosen. If μ^i is a vector such that $\mu^i\eta_i=0$, $\mu_i\lambda^i_{1|}=1$, $\mu^i\lambda_{2|i}=-1$, then it follows that μ_i satisfies the equation of Killing $\mu_{i,j}+\mu_{j,i}=0$ and hence S admits the group of motions. Moreover, the angle between $\lambda^i_{1|}$ and $\lambda^i_{2|}$ is a constant along the vector μ^i. In this manner one may generate on S, which admits the group of motions, a net without detours (=réseau des chemins équax=Netz ohne Umwege). *V. G. Grove* (East Lansing, Mich.).

Strubecker, Karl. Komplexe Geometrie und aufrechte Ellipsenbewegung. Jber. Deutsch. Math. Verein. **50**, 43–58 (1940). [MF 2773]

Krames, Josef. Über die durch aufrechte Ellipsenbewegung erzeugten Regelflächen Θ. Jber. Deutsch. Math. Verein. **50**, 58–65 (1940). [MF 2774]

Darboux has shown that a three dimensional system can be moved so that each of its points describes an ellipse. A special case of these motions, which has been investigated by several authors, is obtained in this way: let a system move so that one of its straight lines coincides with itself, and that one of its points moves on a given plane. Krames labeled this kind of motion as "Aufrechte Ellipsenbewegung" (upright elliptic motion, abbreviated: U.E.M.). In the first of the two papers, Strubecker reports and completes investigations of Study, Gruenwald and Blaschke on close connections between the U.E.M. and the geometry of manifolds of two complex or two dual complex variables. A dual complex quantity has the form $x+\epsilon y$, x and y being real and $\epsilon^2=0$. Study investigated in his "Geometrie der Dynamen" a mapping of the straight lines of the R_3 on the dual complex plane formed by all points with three homogeneous dual complex coordinates, or on a dual complex bundle. He denotes as a "chain" the set of all dual complex lines, which are, together with three given lines, in the same pencil or in the same cone, and have with the given lines a real crossratio. Strubecker proves that these chains correspond to the ruled surfaces generated by a straight line of a system in an U.E.M. A similar connection exists between these ruled surfaces, and "chains" in a complex geometry.

In the second paper, Krames studies these ruled surfaces Θ, generated by an U.E.M. He proves that each surface is contained in the "isotropic congruence" formed by all straight lines on a system of confocal quadrics and he studies a few properties of these surfaces Θ. *E. Helly.*

Weitzenböck, R. Zur projektiven Differentialgeometrie der Regelflächen im R_4. II. Nederl. Akad. Wetensch., Proc. **43**, 548–556 (1940). [MF 3100]

The first communication was published in the same Proc. **43**, 440–448 (1940) [cf. these Rev. **2**, 17]. In the second communication Weitzenböck investigates linear spaces which are associated, in an invariant way, with an arbitrary generatrix A of a ruled surface F in a projective 4-space. Two generatrices A and A_1 of the given ruled surface have a connecting 3-space, whose limit position, if A approaches A_1, defines the tangential space along A. The two tangential spaces along A and A_1 have a plane (a 2-space) in common, which has a limit position if A approaches A_1. Weitzenböck calls this limit position the "attached plane" (Heftebene) of the generatrix A. Three generatrices define one transversal line, whose limit position, when the three generatrices approach A, is called the "attached line" (Heftgrade) of A. The attached line of A meets A in an "attached point" (Heftpunkt). The set of all attached lines of the ruled surface F form the "attached ruled surface" (Heftfläche) of F. The author proves that, in general, the attached surface of the attached surface coincides with F. Weitzenböck derives the symbolic equations of these and several other associated linear spaces and analyzes relations between them. *Helly.*

Weitzenböck, R. Zur projektiven Differe[illegible]trie der Regelflächen im R_4. III. Nederl. Ak[illegible]sch., Proc. **43**, 668–673 (1940). [MF 3105]

[Cf. the preceding review.] The author [illegible]s the investigation of the linear spaces associated wi[illegible] arbitrary generatrix A of a ruled surface F in 4-space, and the study of the related invariants. In an earlier paper [the same Proc. **43**, 325–333 (1940); these Rev. **1**, 270] the author has shown how, if four lines are given, a fifth "associated" line can be found, and how the limit position of this line can be determined if the four given lines are genera-

trices of F and approach one of them. In this paper he resumes the investigation of this associated line G ("Beigrade") of the line A, and proves that the line G meets the attached plane of A. *E. Helly* (Paterson, N. J.).

Weitzenböck, R. Zur projektiven Differentialgeometrie der Regelflächen im R_4. IV. Nederl. Akad. Wetensch., Proc. **43**, 797–804 (1940). [MF 3179]

[Cf. the preceding review.] The author presents, in symbolic form, the simplest differential invariants of a given ruled surface F in 4-space. Let the coordinates $a_{i\alpha}$ of a generatrix A of F be functions of a parameter t. If t is replaced by $\bar{t}$ so that $t=\varphi(\bar{t})$, then each differential invariant of F is multiplied by a factor $(\phi'(\bar{t}))^{\rho}$. The exponent ρ may be called ϕ'-weight of the invariant. The same invariant is multiplied by a factor λ^{σ} if all coordinates $a_{i\alpha}(t)$ are multiplied by the same factor $\lambda(t)$. The exponent σ is called λ-weight of the invariant. Weitzenböck proves that the simplest differential invariants are an invariant Q with weights 9 and 5, and two invariants A and B of ϕ'-weight 20 and λ-weight 10. *E. Helly* (Paterson, N. J.).

Weitzenböck, R. Zur projektiven Differentialgeometrie der Regelflächen im R_4. V. Nederl. Akad. Wetensch., Proc. **43**, 805–814 (1940). [MF 3180]

[Cf. the preceding review.] The derivative with respect to the parameter t of an invariant or covariant of a ruled surface F in 4-space is, as a rule, not invariant. Weitzenböck introduces, in different ways, "covariant derivatives" which may replace the ordinary derivatives, and he uses these covariant derivatives in order to give an invariant expression of a general generatrix of F. He shows that this problem is closely connected with the problem of finding a fundamental system $J_1, J_2, \cdots, J_m$ of differential invariants, so that every other differential invariant of F can be expressed by these invariants J_s and by their derivatives dJ_1/dt, dJ_2/dt, $\cdots$. Weitzenböck gives a fundamental system of 7 invariants and shows that the ruled surface is determined, except for projective transformations of the 4-space, when these 7 differential invariants are given as functions of the parameter t. This system can be reduced to five functions if the parameter t and the common factor λ of the coordinates $a_{i\alpha}$ are normed in a certain way. *E. Helly* (Paterson, N. J.).

Behari, Ram. A theorem on normal rectilinear congruences. Proc. Indian Acad. Sci., Sect. A. **12**, 205–207 (1940). [MF 2951]

The author considers a certain family of ruled surfaces on a normal rectilinear complex, namely the two ruled surfaces through each line of the complex whose curves of striction lie on the middle surface of the complex. [See Eisenhart, Differential Geometry, p. 422.] These surfaces are called by Eisenhart "ruled mean surfaces." It is known that, if we form the spherical representation of the complex on the unit sphere S, then the curves on S corresponding to the ruled mean surfaces form an orthogonal system. In the present paper, which is actually an abstract of his work, the author shows that, if the congruence is normal, the curves on S form a system of isometric curves. *J. W. Green.*

Dwinger, Ph. Ueber ein System von drei Strahlenkongruenzen. Nederl. Akad. Wetensch., Proc. **43**, 1023–1031 (1940). [MF 3534]

The author replaces the classical notations by Study's dual numbers. Thus a straight line A is determined by a dual unitary vector $A=a+\epsilon\bar{a}$ with $\epsilon^2=0$ and $a^2=1$; here the vector a gives the direction, and the vector $\bar{a}$ is the momentum of a unitary vector of the line A with respect to the origin. If a, $\bar{a}$ are functions of two parameters u, v, the line A generates a congruence K, the fundamental dual differential form of which is

$$dA^2=(edu^2+2fdudv+gdv^2)+\epsilon(\bar{e}du^2+2\bar{f}dudv+\bar{g}dv^2).$$

Now to every line A of K the author makes correspond two new straight lines B, C. They are called principal normals of K at A (Hauptnormalen, or Kehlspunktnormalen zu den Hauptflächen durch den Strahl A). The lines A, B, C are the edges of a rectangular trihedral. The lines B, C generate two new congruences K', K''. The author finds necessary and sufficient conditions in order that the relationship among these congruences be involutory, that is to say, that K and K' [K and K''] be congruences of the principal normals of the congruence K'' [K']. *G. Fubini.*

Schneidt, Max. Von Translationsflächen erzeugte Ribaucoursche Strahlensysteme. Monatsh. Math. Phys. **49**, 109–123 (1940). [MF 2991]

Let F and F_1 be the focal points of a straight line belonging to a rectilinear congruence K, and let u, v be the parameters of the developable surfaces of K. The points F and F_1 are functions of u, v and generate the two focal surfaces of K; we suppose that

$$Edu^2+2(EF)^{\frac{1}{2}}\cos\epsilon dudv+Gdv^2$$

and

$$E_1du^2+2(E_1F_1)^{\frac{1}{2}}\cos\epsilon_1dudv+G_1dv^2$$

are the Gaussian linear elements of these surfaces. We can suppose that

$$\frac{F_u}{G^{\frac{1}{2}}}=-\frac{F_{1v}}{G_1^{\frac{1}{2}}},$$

are the direction cosines of the straight line FF_1. If the lines $v=$ const. on S and $u=$ const. on S_1 are lines of shadow (Schattengrenzlinien, according to Blaschke), the unity-vector, whose direction cosines are $(1/G^{\frac{1}{2}})F_v$, will be a function only of v, and the unity-vector $(1/E_1^{\frac{1}{2}})F_{1u}$ will be a function only of u. In this case the author says that K is a congruence (S). He determines all these congruences and finds that they are closely related to the surfaces of translation. He studies the conditions that such a congruence (S) is normal, and finds that, in this case, its straight lines are normal to a surface whose lines of curvature are plane. He generalizes this result to the surfaces which possess only a system of plane lines of curvature. *G. Fubini.*

Fubini, Guido. On Bianchi's permutability theorem and the theory of W-congruences. Ann. of Math. (2) **41**, 620–638 (1940). [MF 2556]

This paper concerns itself with a type of transformation of surfaces. Two surfaces are said to be congruence-transforms of each other if they are in one-to-one point correspondence and are the focal surfaces of the congruence of lines joining corresponding points. Let the points x, y generate surfaces each of which is a congruence-transform of each of three other surfaces generated by points z, t, η. It is proved that if one of the six congruences of lines joining the corresponding points is a W-congruence, the other five are also W-congruences. If, moreover, there exists not only surfaces generated by x and y but also ∞^1 surfaces which are congruence-transforms of the surfaces generated by x, y, then all congruences joining corresponding points are

W-congruences. This is Bianchi's permutability theorem. Attention is paid to certain converse theorems. For example, the paper solves the problem: Under what conditions is it possible for two surfaces, each a congruence-transform of a surface, to be congruence-transforms of at least two other surfaces, and consequently of a ∞^1 of surfaces? The condition is given in analytic form. Again ∞^1 congruences are said to constitute a pencil if they have a common first focal surface, if the corresponding points of the second focal surface generate a line, and if the tangent planes to the second focal surfaces belong to a pencil. It is proved that if one congruence of a pencil is a W-congruence, all of the congruences of the pencil are W-congruences. Two W-congruences always determine a pencil of congruences.

V. G. Grove (East Lansing, Mich.).

Salini, Ugo. Intorno alla superficie Ω di Demoulin. Ann. Mat. Pura Appl. (4) **19**, 125–140 (1940). [MF 3048]

Let u, v be the parameters of the lines of curvature of a surface S, and let K be the congruence of their normals. On every straight line of K (or on every normal to S) we choose a point M, which will generate another surface Σ. Is it possible to choose M in such a way that the net u, v on Σ is a conjugate net with equal invariants? The analogous problem for non-normal congruences K becomes a problem of projective differential geometry. In the first case S is a surface Ω. Demoulin and Calapso demonstrated that these surfaces are Lie-transforms of surfaces R. The metrical problem was studied by Demoulin, the projective problem by B. Segre. The author compares various results and calculations of Demoulin, B. Segre and Calapso. Finally he studies the surfaces S for which the point M (and the corresponding net Σ) depends on an arbitrary parameter.

G. Fubini (Princeton, N. J.).

Cartan, E. Sur une classe de surfaces apparentées aux surfaces R et aux surfaces de Jonas. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 72–78. [MF 3505]

Let us consider a net (conjugate net) in the usual space. This net defines two Laplace equations for the points and for the tangent planes, respectively. Their invariants h, k, $\bar{h}$, $\bar{k}$ are connected by the equations $h=k$ and $\bar{h}=\bar{k}$, or $h=\bar{k}$ and $k=\bar{h}$, or $h=\bar{h}$ and $k=\bar{k}$. In the first case the corresponding surfaces are the surfaces J of Jonas; in the second case these surfaces are R; the third case is new; the author calls the corresponding surfaces E. By means of Pfaffian forms the author proves that the surfaces J (like the surfaces R) also depend on six functions of one parameter, and that the surfaces E depend only on five functions of one parameter. The author also gives a method for finding the nets J, R, E which can lie on a given surface.

G. Fubini (Princeton, N. J.).

Finikoff, S. Réseaux de Rozet. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 151–180 (1940). (Russian. French summary) [MF 2734]

[Review written from summary only.] If M_1, M_2 are the focal points of a rectilinear congruence K, and if u, v are its developables, the lines u, v on the focal surfaces are conjugate. Therefore M_2 can be deduced from M_1 (and M_1 from M_2) by means of Laplace's transformations. It is well known that we can construct a Laplace succession $\cdots$, M_5, M_3, M_1, M_2, M_4, M_6, $\cdots$ such that every point M can be deduced from the preceding (and from the following) by means of a Laplace transformation related to the parameters u, v.

The congruence K is called a Rozet's congruence in the direction M_1M_2 if the points M_1, M_3, M_5, M_4 belong to the same plane. The author proves that these congruences depend on one function of two variables, and that a Rozet's congruence is W if and only if it belongs to a linear complex. He studies also many other particular congruences: the congruences K such that the congruence generated by the lines M_1M_3 is W, the congruences K which are Rozet's congruences in both directions M_1M_2 and M_2M_1, the congruences K such that the corresponding congruence generated by the lines M_2M_4 is also a Rozet's congruence, and so on.

G. Fubini (Princeton, N. J.).

Dubnov, J. et Fuchs, S. Sur quelques réseaux de l'espace analogues au réseau de Tchebychev. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 102–105 (1940). [MF 3597]

By a net in the Euclidean E_3 is meant a system of three congruences K_i of curves C_i ($i=1, 2, 3$). Through any point A there pass three curves C_i, and it is supposed that their tangents are not coplanar. The system is called holonomic if there exist three systems Σ_i of ∞^1 surfaces S_i such that each K_i is generated by the intersections of a surface S_j with a surface S_k. These intersections are therefore the curves C_i. The net is called strongly parallel if the directions of the curves C_i along any curve C_j are parallel; in a similar way the authors define the weakly parallel nets; they also define some nets which are the generalization of Tschebycheff's nets. They find the necessary and sufficient conditions in order that a given net belongs to one of the preceding classes of nets. Among the theorems we mention: Every strongly parallel net is holonomic (but not conversely).

G. Fubini (Princeton, N. J.).

Weise, Karl Heinrich. Invariante Charakterisierung von Kurvennetzen. Math. Z. **46**, 665–691 (1940). [MF 3375]

This paper studies properties of a net defined by

$$\varphi_{\alpha\beta}du^\alpha du^\beta=0, \qquad |\varphi_{\alpha\beta}|\neq 0, \tag{1}$$

in a Riemannian space of two dimensions with $ds^2=g_{\alpha\beta}du^\alpha du^\beta$. To every root λ of the characteristic equation $|\varphi_{\alpha\beta}-\lambda g_{\alpha\beta}|=0$ there corresponds a vector v with covariant components v_α satisfying

$$g_{\alpha\beta}-v_\alpha v_\beta=\frac{1}{\lambda}\varphi_{\alpha\beta}. \tag{2}$$

One has $\lambda\neq 0$ since $|\varphi_{\alpha\beta}|\neq 0$. If we multiply (2) by an arbitrary factor, the v_α do not change. The vector v defines the net; and conversely the net determines the vectors v and w corresponding to the two characteristic roots λ and μ. Disregarding the trivial case $\lambda=\mu$, the author finds the components w_α of w as linear combinations of the v_α and conversely. He also finds the unitary vectors tangent to the curves of the net as (very simple) linear combinations of the vectors v, w. In order to study the net it is sufficient to study one of the vectors v, w; therefore one studies the covariant derivatives $v_{\alpha,\beta}$ of the v_α, writing them as linear combinations either of v_α, w_α, or of v_1, v_2. By means of these equations one arrives at the "geodetic curvature" and at the "Tschebycheff-curvature" of the curves of the net. The integrability conditions lead the author to some geometrical results. Using the well-known transformation of curvilinear integrals into surface integrals, the author finds many generalizations of the classical theorem for the integral of the geodetic curvature of a closed curve, and other analogous theorems. Finally he investigates many special cases.

G. Fubini (Princeton, N. J.).

König, Robert und Weise, Karl Heinrich. Zur konformen Abbildung zweier Flächen mit beliebigen Parametern. Ber. Verh. Sächs. Akad. Wiss. Leipzig **92**, 19–32 (1940). [MF 3487]

The determination of differential conditions under which the mapping of one surface upon another is conformal, when both are given in terms of the same pair of parameters (u^1, u^2), is a classical result of Gauss. The authors investigate the conditions of conformality in the more general case when the surfaces are given in terms of arbitrary parameters (u^1, u^2) and (v^1, v^2) between which a differential mapping $v^\alpha = v^\alpha(u^1, u^2)$, $\alpha = 1, 2$, is given. The conditions are given in tensorial form, showing their invariance under transformations of parameters; they naturally include as special cases the Beltrami and Cauchy-Riemann equations. Analogous to the Laplace equation, which is an integrability condition for the Cauchy-Riemann equations, an integrability equation is established for the general case. This latter equation is shown to be the Euler equation for a variation problem which might be considered to be a generalization of the Dirichlet problem.

E. F. Beckenbach (Ann Arbor, Mich.).

Weise, Karl Heinrich. Bemerkung zur Abbildung zweier Flächen. Ber. Verh. Sächs. Akad. Wiss. Leipzig **92**, 45–50 (1940). [MF 3489]

In continuation of the work of the preceding paper, the author, no longer assuming conformality, determines the Tissot indicatrix for the map in the general case. This gives a description of the map deformation in the small. Conformality is treated as a special case, with results consistent with those of the preceding paper. *E. F. Beckenbach.*

Ermolaev, L. Image projective d'une surface. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 735–737 (1940). [MF 3574]

Let two surfaces S and Σ be in one-to-one point correspondence, the point M_0 on S and the point M_3 on Σ corresponding. Suppose that neither S nor Σ is a focal surface of the congruence of lines M_0M_3. Let M_1, M_2 be two points on the intersection of the tangent planes to S and Σ at M_0 and M_3, respectively. The coordinates of M_i satisfy differential equations of the form

$$dM_i = (a_i^k du + b_i^k dv) M_k,$$

wherein

$$\frac{\partial a_i^k}{\partial v} - \frac{\partial b_i^k}{\partial u} = a_j^k b_i^j - a_i^j b_j^k.$$

These equations are simplified by choosing the asymptotic curves on S as parametric and by choosing M_1, M_2 on these asymptotic tangents. Consider a line l in the pencil of tangent lines at M_0 to S. To this line there corresponds a tangent λ to Σ at M_3 in the point correspondence between these surfaces. Let λ intersect M_1M_2 in P. Designate by l' the line M_0P. The pencil of lines l and the pencil l' are projectively related. The author calls this projectivity "the projective point-image of Σ on S."

If $du : dv$ is the direction of l and $\delta u : \delta v$ the direction of l', then du, dv, δu, δv satisfy the equation

(P) $$a_3^2 du\delta u + b_3^2 dv\delta u - a_3^1 du\delta v - b_3^1 dv\delta v = 0.$$

By dual considerations another projectivity may be established among the tangent lines to S at M_0. This projectivity is called "the projective tangential image of Σ on S," and is given by

(T) $$a_1^0 du\delta u + b_1^0 dv\delta u - a_2^0 du\delta v - b_2^0 dv\delta v = 0.$$

Using (P) and (T) it is shown that the asymptotic curves on S and Σ correspond in the point correspondence between them if and only if the line l' which corresponds to $du : dv$ in (P) and the line l'' which corresponds to $-du : dv$ in (T) separate the asymptotic tangents at M_0 harmonically. In case the asymptotic tangents to S at M_0 correspond in (P), (P) is an involution and the double lines of (P) separate the asymptotic tangents harmonically. The special case in which Σ is the envelope of the quadric of Lie of S is considered, and a new characterization of the directrices of Wilczynski is given. *V. G. Grove* (East Lansing, Mich.).

Ermolaev, L. Quelques classes de correspondances ponctuelles de surfaces, déterminées par les images projectives. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 422–424 (1940). [MF 3217]

The author continues the discussion of the projectivities (P) and (T) in the paper reviewed above. The lines l' and l'' which correspond to the same line l in (P) and (T) generate projectively related pencils of lines. This projectivity is never an involution. The double lines of this projectivity are tangent to the curves of a conjugate net. A conjugate net is called a net of Peterson if the net which corresponds to it is also a conjugate net. It is shown that if the double lines of (P) are the tangents to the curves of the net of Peterson, then the double lines of (T) are the tangents to the curves of the same net. Conversely if the double lines of (P) and (T) coincide and if the curves to which they are tangent form a conjugate net, this net is a net of Peterson and S and Σ are in the relation of a fundamental transformation F of Eisenhart. *V. G. Grove.*

Efimoff, N. Déformation du voisinage d'un point parabolique d'une surface. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 134–136 (1940). [MF 3547]

Let there be given the equation of a surface U in the form

$$z = \tfrac{1}{2}(ax^2 + 2bxy + cy^2) + \tfrac{1}{6}(a_1x^3 + \cdots).$$

Let $O(0, 0)$ be a parabolic point on U with not all a, b, c zero. Let $S(u, v)$ be a manifold of two dimensions with the metric $ds^2 = Edu^2 + 2Fdudv + Gdv^2$, wherein E, F, G are regular at $O(0, 0)$. Let $K(u, v)$ be the Gaussian curvature of S and $K(0, 0) = 0$, but $K(u, v) \not\equiv 0$. The metric on S is said to be singular if among the geodesics on S through O there exist two along which $K(u, v) = 0$, $dK = 0$. It is proved that in a regular deformation of U with a determinate index at its parabolic point [cf. N. Efimoff, Rec. Math. [Mat. Sbornik] N.S. **6** (**48**), 427–474 (1939); these Rev. **2**, 19] the value of the index is invariant under the deformation. And if the metric of S is not singular, then all surfaces U applicable to S and with determinate indices at O have the same index at O. *V. G. Grove* (East Lansing, Mich.).

Efimoff, N. Démonstration de l'existence d'une surface localement non déformable. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 314–317 (1940). [MF 3244]

This note concerns itself with establishing criteria that a surface be non-deformable in the neighborhood of a parabolic point of order $p \geqq 3$. Let the equation of the surface be written in the form

$$z = F^m(x, y) + F^{m+1}(x, y), \qquad m \geqq 4,$$

wherein $F^i(x, y)$ are homogeneous polynomials of degree i

and where in particular

$$F^m(x, y)=a_0x^m+a_1\binom{m}{1}x^{m-1}y+\cdots+a_my^m.$$

Let

$$\Phi(xy)=b_0x^n+b_1x^{n-1}y+b_2x^{n-2}y^2+\cdots+b_nx^n.$$

Let $N(F^m)$ be the maximum value of l such that

$$F^m_{xx}\Phi^n_{yy}-2F^m_{xy}\Phi^n_{xy}+F^m_{yy}\Phi^n_{xx}\equiv 0$$

implies that $\Phi\equiv 0$ for $n=m+1, m+2, \cdots, m+l$. Let A represent the matrix

$$\begin{pmatrix} a_0 a_1 \cdots a_{m-2} \\ a_1 a_2 \cdots a_{m-1} \\ a_2 a_3 \cdots a_m \end{pmatrix}.$$

Then, if (1) the real factors of $F^m(x, y)$ have multiplicity no greater than $m-2$; (2) $F^m(x, y)$, $F^{m+1}(x, y)$ do not possess common multiple real linear factors; (3) $N(F^m)=\infty$; and (4) A is of rank 3, the given surface is non-deformable in the neighborhood of the parabolic point. In particular the surface $z=x^8y^6+x^{10}+y^{10}$ satisfies the conditions of the theorem in the neighborhood of the origin. *V. G. Grove.*

Coburn, N. A characterization of Schouten's and Haydens' deformation methods. J. London Math. Soc. **15**, 123–136 (1940). [MF 2522]

This paper is an interpretation of the method of absolute variation [see, e.g., Schouten-Struik, Einführung in die neueren Methoden der Differentialgeometrie, vol. I, p. 149] and that of direction deformation [H. A. Hayden, Proc. London Math. Soc. (2) **37**, 416–440 (1934); see also P. Dienes-E. T. Davies, J. Math. Pures Appl. (9) **16**, 111–150 (1937)]. It is shown that Schouten's deformation of affinors is completely determined by the fact that the coordinate differential vectors in L_n and the unit affines of L_n are deformed by parallelism in the local affine space E_n; Hayden's deformation by the fact that the measure vectors of L_n are deformed by parallelism in L_n, and the vectors of L_m (in L_m coordinates) and the unit affinor of L_n are deformed by parallelism in the E_n. Here X_m is a submanifold of the manifold L_n with general linear connection. The paper concludes by studying the relation to a variation problem of W. Slebodzinski [Prace Mat.-Fiz. **39**, 55–62 (1932)].

D. J. Struik (Cambridge, Mass.).

Wagner, V. Differential geometry of non-linear non-holonomic manifolds in the three-dimensional Euclidean space. Rec. Math. [Mat. Sbornik] N.S. **8 (50)**, 3–39 (1940). (English. Russian summary) [MF 3186]

The present paper is a specialization of a previous one, "A generalization of non-holonomic manifolds in Finslerian space" [Abh. Tschernyschewsky Univ. Saratow. Serie Phys.-Math. Forschungsinstituts **1 (14)**, no. 2, 67–97 (1938)], to the case where the ambient is Euclidean 3-space. The development is independent of the previous paper and facilitates geometric interpretation. It is connected with the geometrization of the theory of the Monge equation first considered by Lie and recently developed by D. Sinstow [1937]. Classical vector notation is used throughout.

The general non-linear, non-holonomic manifold C^2_3 in Euclidean 3-space consists of a field of conical pencils of directions. These form a 4-space X_4 with coordinates (x, y, z, u), where u is the parameter generating the cone at (x, y, z). In X_4 exist curves, surfaces and hypersurfaces corresponding to curves and surfaces in C^2_3 whose tangential elements belong to the field of cones. They will be integral varieties of certain Pfaffian systems in X_4. The field of cones permits a definition of parallelism in C^2_3 which leads to other differential-geometric notions: lines of curvature, geodesics, geodesic surfaces, etc., which in turn correspond to specialized Pfaffians in X_4. The main results of the paper rest upon this interplay between Pfaffians in X_4 and geometry in C^2_3.

J. L. Vanderslice (Bethlehem, Pa.).

Haimovici, Mendel. Sur la définition intrinsèque d'un élément plan normal à une variété non holonome. C. R. Acad. Sci. Paris **210**, 595–596 (1940). [MF 3032]

An application of a general method due to Cartan [Ann. École Norm. (3) **25**, 57–73 (1908)] permits the author to consider non-holonomic varieties in a Riemann space from a new intrinsic point of view and to formulate a definition of the normal flat to the variety. *A. Fialkow.*

Rosenson, N. Sur les espaces riemanniens de classe I. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 181–192 (1940). (Russian. French summary) [MF 2735]

A Riemannian manifold of n dimensions V_n is said to be of class 1 if its fundamental form $ds^2=g_{\alpha\beta}dx^\alpha dx^\beta$, $\alpha, \beta=1, \cdots, n$, can be written in the form $ds^2=\sum(dy^i)^2$, $i=1, 2, \cdots, n+1$. A necessary condition for the V_n is derived in the case of even n and of odd n. In the case of even n this condition is

$$^{(n)}W^{\cdots\gamma\delta}_{\alpha\beta}=\frac{^{(n)}W_1}{n(1-n)}(\delta^\gamma_\alpha\delta^\delta_\beta-\delta^\delta_\alpha\delta^\gamma_\beta),$$

where

$$^{(s)}W_1=\,^{(s)}W^{\cdots\beta\alpha}_{\alpha\beta}, \qquad s=2, 3, \cdots, n,$$

$$^{(s)}W^{\cdots\gamma\delta}_{\alpha\beta}=2^{(s/2)-1}\Big[^{(s-1)}U^\gamma_\alpha\Omega^\delta_\beta+\,^{(s-2)}U^\gamma_\alpha\Omega^{\cdot\delta}_{\beta\cdot}+\cdots+\,^{(1)}U^\gamma_\alpha\underbrace{\Omega^{\cdots\delta}_{\beta\cdots}}_{s-2}$$

$$-\,^{(s-1)}U^\delta_\alpha\Omega^\gamma_\beta-\,^{(s-2)}U^\delta_\alpha\Omega^{\cdot\gamma}_{\beta\cdot}-\cdots-\,^{(1)}U^\delta_\alpha\Omega^{\cdots\gamma}_{\beta\cdots}\Big],$$

$$^{(s)}U^\delta_\alpha=\underbrace{\Omega^{\cdots\delta}_{\alpha\cdots}}_{s-1}+p_1\underbrace{\Omega^{\cdots\delta}_{\alpha\cdots}}_{s-2}+\cdots+p_{s-2}\Omega^{\cdot\delta}_{\alpha\cdot}+p_{s-1}\Omega^\delta_\alpha,$$

$$\underbrace{\Omega^{\cdots\alpha_\delta}_{\alpha_1\cdots}}_{s-2}=\Omega^{\alpha_3}_{\alpha_1}\Omega^{\alpha_4}_{\alpha_2}\cdots\Omega^{\alpha_\delta}_{\alpha_{s-1}},$$

$p_i=(-1)^i\sum$ products of i factors $\Omega_{\alpha\alpha}$; $\Omega_{\alpha\beta}$ is the second fundamental tensor of the V_n, referred to principal axes and related to $R^{\cdots\gamma\delta}_{\alpha\beta}$ by means of the Gauss equations $R_{\alpha\beta\gamma\delta}=\Omega_{\alpha\gamma}\Omega_{\beta\delta}-\Omega_{\alpha\delta}\Omega_{\beta\gamma}$. The tensor $^{(s)}W^{\cdots\gamma\delta}_{\alpha\beta}$ is of degree $n/2$ in the curvature tensor $R^{\cdots\gamma\delta}_{\alpha\beta}$. For the odd case a similar formula is derived, now with a tensor of degree n in $R^{\cdots\gamma\delta}_{\alpha\beta}$. Reference is made to papers by K. H. Weise [Math. Ann. **110**, 522–570 (1935)] and T. Y. Thomas [Acta Math. **67**, 169–211 (1936)]. *D. J. Struik* (Cambridge, Mass.).

Usunoff, Nicolas. Über das vierdimensionale Problem der Riccikurven im Riemannschen Raum. Monatsh. Math. Phys. **49**, 124–152 (1940). [MF 2992]

Consider a four-dimensional Riemannian space R_4 with coordinate system x^i $(i, j=1, \cdots, 4)$. When functions $q_\alpha{}^i$ $(\alpha, \beta=1, \cdots, 4)$ of position are given, the differential equations

$$dx^1/q_\alpha{}^1=\cdots=dx^4/q_\alpha{}^4$$

define a system of four congruences. It is assumed that the $q_\alpha{}^i$ are such that they determine a 4-tuple of orthogonal vectors at any point of R_4 and hence the system of congruences are mutually orthogonal. An equivalent scheme

for determining the same system of mutually orthogonal congruences is given by means of the vectors $q_\alpha{}^i=\lambda_\alpha q_\alpha{}^i$ (not summed on α). If this system of congruences lies in the principal directions of some Ricci tensor associated with a Riemannian space $\bar{R}_4$, then the congruences are said to be Ricci curves of $\bar{R}_4$. The author's problem is: given a Riemannian space R_4 and a system of mutually orthogonal congruences $q_\alpha{}^i$, what conditions must be satisfied by multipliers λ_α in order that the vectors $q_\alpha{}^i$ of $\bar{R}_4$ form Ricci curves? Using Cartan's symbols, the author finds that the λ_α must satisfy a complicated system of second order partial differential equations [equation 33]. The coefficients in this system are the non-holonomic geometric objects $\bar{C}^\alpha_{\beta\gamma}$ of $\bar{R}_4$ which are related to the $C^\alpha_{\beta\gamma}$ of R_4 [equation 32]. To simplify the calculations, the author restricts himself to a study of the case where the Ricci curves are geodesics of $\bar{R}_4$ ($\bar{C}^\alpha_{\beta\gamma}=0$). In this case, by studying the integrability conditions due to the additional assumption and classifying the $\bar{C}^\alpha_{\beta\gamma}$, the λ_α [equation 33] are shown to satisfy systems of first order differential equations. Reviewer's note: The paper is difficult to read because of ambiguity in notation. The proof on page 131 is wrong although the final result is correct. *N. Coburn* (Austin, Tex.).

Lusternik, L. Les géodésiques fermées sur les multiplicités sphériques à plusieurs dimensions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 320–322 (1940). [MF 3555]

The author sketches a proof that any Riemannian manifold which is the analytic homeomorph of an n-sphere has on it either $(2n-1)$ closed geodesics of different lengths or a family of such geodesics which is of the power of the continuum. The discussion is restricted to the case $n=3$. If S_3 denotes the unit sphere $\sum x_i{}^2=1$ in E_4, it is indicated how the space A_3 of circles on S_3 contains classes A_i of cycles of various dimensions less than 7 which are not homologous to zero. If the circles on S_3 are deformed continuously such that the lengths of the corresponding curves on the Riemannian manifold M vary continuously, the class A_i deforms into a family B_i of closed curves; let C_i denote the inferior limit for the totality of deformations of the maximum of the lengths of the curves of M corresponding to those of B_i. The non-vanishing constants C_i are 5 in number and to each there corresponds a closed geodesic of length C_i. Thus if $C_i\neq C_j$, $i\neq j$, there exist five closed geodesics of different lengths. It is shown that if two of the C_i are equal, there exists a family of closed geodesics of the power of the continuum. As the author states, Morse [The Calculus of Variations in the Large, New York, 1934] has proved the existence of $n(n+1)/2$ closed geodesics in the case under consideration. Under certain conditions stated by Morse, no one of the closed geodesics is a covering of any other. That hypothesis implies the same result for the $(2n-1)$ geodesics of the present paper. *G. A. Hedlund.*

Wong, Yung-Chow. On a certain matrix occurring in the theory of helices. J. London Math. Soc. **15**, 168–172 (1940). [MF 3134]

This matrix, of rank k, is

$$M=\|Dj^s, D^{(2)}j^s, \cdots, D^{(n)}j^s\|,$$

where j^s is a unit vector defined along the curve $\xi^s=\xi^s(s)$ in a Riemannian V_n, $D=i^s\nabla_s$, $D^{(2)}=DD$, etc., and $i^s=d\xi^s/ds$. The curvatures $\kappa_1, \kappa_2, \cdots, \kappa_{h-1}$ are constant, the other κ_i are zero, $h\leqq n$. Then the rank k is either equal to h or equal to $h-1$. Both cases are analyzed. Reference is made to work by J. H. Butchart [Amer. J. Math. **55**, 126–130 (1933)]. *D. J. Struik* (Cambridge, Mass.).

Wong, Yung-Chow. On the principal directions of a tensor. J. London Math. Soc. **15**, 172–183 (1940). [MF 3135]

Given a symmetric tensor $b_{\alpha\lambda}$ in a Riemannian V_n with its set of m mutually orthogonal real principal R_α, $\alpha=n_s$, $\sum n_s=n$, none of which is tangent to the null cone of V_n. It is shown that the smallest number of principal directions of $b_{\alpha\lambda}$, on which the direction of a given vector is linearly dependent, is equal to the rank of certain matrices formed with the aid of the tensors

$$b_{\alpha\lambda}=b^r_{\alpha}\,b_{r\lambda}, \quad b_{0\alpha\lambda}=a_{\alpha\lambda}\ \text{(fundamental tensor)}.$$

This leads to a comparison of the principal directions of these tensors and to conditions for the coincidence of the principal curvature directions and the Ricci directions of a proper V_{n-1} in an S_n of constant curvature, connected with results of A. Fialkow [Ann. of Math. (2) **39**, 762–786 (1938)]. *D. J. Struik* (Cambridge, Mass.).

Cisotti, Umberto. Una notevole scomposizione dei tensori. Boll. Un. Mat. Ital. (2) **2**, 210–215 (1940). [MF 2974]

A tensor of valence m, of ordinary Euclidean space, can be decomposed, for m even, into an isotropic tensor and another tensor with zero linear invariant, and for n odd into a semi-isotropic tensor and another tensor with zero linear semi-invariant. The isotropic or semi-isotropic tensor is in both cases of minimal "deviation." Reference is made to papers by Biggiogero [Atti Accad. Naz. Lincei. Rend. (6) **17**, 611–616 (1933)] and Oberti [Ist. Lombardo, Rend. (2) **69**, 397–402 (1936)]. As example we take

$$T_{ik}=A\delta_{ik}+T'_{ik}, \quad \sum_{i=1}^{3} T'_{ii}=0, \quad A=\tfrac{1}{3}\sum_{i=1}^{3} T_{ii},$$

$$T_{ikl}=B\epsilon_{ikl}+T''_{ikl}, \quad T''_{[123]}=0, \quad B=T_{[123]}.$$

D. J. Struik (Cambridge, Mass.).

Urabe, Minoru. Geometry of parallel displacement making a volume invariant. J. Sci. Hirosima Univ. Ser. A. **10**, 95–108 (1940). [MF 3055]

Let X_n be an n-dimensional space on whose points are defined, independently of each other, a Riemannian metric tensor $g_{\lambda\mu}$ and arbitrary coefficients of connection $\Gamma^\nu_{\lambda\mu}$. General spaces like X_n and the displacement of vectors in them have previously been studied by J. A. Schouten [Der Ricci-Kalkül, 1924, pp. 62–75, 89–90]. In this paper the author defines a scalar V which is analogous to, but not identical with, the volume spanned by m ($m\leqq n$) vectors in X_n. It is shown that the necessary and sufficient condition that V be invariant under the parallel displacement defined by the $\Gamma^\nu_{\lambda\mu}$ is that the $\Gamma^\nu_{\lambda\mu}$ have a particular form which is found explicitly. Various special cases are discussed. *A. Fialkow* (Brooklyn, N. Y.).

Coburn, N. A note on conformal geometry. Proc. Nat. Acad. Sci. U. S. A. **27**, 57–60 (1941). [MF 3653]

The well-known equations for the transformation of the components of the affine connections of two conformally equivalent Riemannian spaces are written down and the question is raised as to whether these equations are sufficient for the conformal equivalence of two Riemannian spaces. The conclusion is reached that the equations in

question together with a condition on the principal directions in the spaces are sufficient for conformal equivalence. *T. Y. Thomas* (Los Angeles, Calif.).

Yano, Kentaro. Concircular geometry. I. Concircular transformations. Proc. Imp. Acad. Tokyo **16**, 195–200 (1940). [MF 2956]

A concircular transformation is defined by the author as a conformal transformation of a Riemann space V_n which changes every geodesic circle of V_n into a geodesic circle. (Here "geodesic circle" means a curve in V_n whose first curvature is constant and whose second curvature is identically zero.) In this sense, the geometry of concircular transformations (concircular geometry) is a generalization of inversive geometry. A discussion of the differential equations and existence questions of concircular geometry appears in the papers of H. W. Brinkmann [Math. Ann. **94**, 119–145 (1925)] and the reviewer [Trans. Amer. Math. Soc. **45**, 443–473 (1939)]. By a straightforward calculation, the author shows that the tensors

$$R^{h}_{ijk}-(R/n(n-1))(g_{ij}\delta^{h}_{k}-g_{ik}\delta^{h}_{j})$$

and

$$R_{ij}-(R/n)g_{ij}$$

are invariant under concircular transformations, where the δ^{h}_{j} are the Kronecker deltas and g_{ij}, R^{h}_{ijk} are the metric tensor and Riemann curvature tensor of V_n, respectively, and $R_{ij}=R^{h}_{ijh}$, $R=R_{ij}g^{ij}$. By means of these concircular tensors, it is shown that (1) a space of constant curvature or (2) an Einstein space is transformed into (1) a space of constant curvature or (2) an Einstein space, respectively, by a concircular transformation. These theorems as well as their converses are also proved in the course of a forthcoming paper by the reviewer entitled "The conformal theory of curves." *A. Fialkow* (Brooklyn, N. Y.).

Yano, Kentaro. Concircular geometry. II. Integrability conditions of $\rho_{\mu\nu}=\phi g_{\mu\nu}$. Proc. Imp. Acad. Tokyo **16**, 354–360 (1940). [MF 3477]

[Continuation of the paper reviewed above.] Let $\bar{g}_{ij}=\rho^2 g_{ij}$ define a concircular transformation between two Riemann spaces. Then it is proved that (1) the hypersurfaces $\rho=$ constant are totally umbilical, (2) the orthogonal trajectories of these hypersurfaces are geodesics, (3) the orthogonal trajectories of these hypersurfaces are tangent to the principal Ricci directions of the Riemann spaces. It is also shown that (4) if a Riemann space contains a one parameter family of totally umbilical hypersurfaces whose orthogonal trajectories are geodesic Ricci curves then the Riemann space admits a concircular transformation. These theorems are related to a number of results in a paper by the reviewer [Trans. Amer. Math. Soc. **45**, 443–473 (1939)]. Thus (1) is an immediate consequence of Theorem 9.2 and the two sentences beginning with equation (8.21) in the reviewer's paper. Similarly, (2) is a corollary of Theorem 9.1 and the two sentences referred to above. All the theorems stated in the present paper may be derived more simply from the sentence containing equation (12.10) of the reviewer's paper by a straightforward calculation. *A. Fialkow.*

Yano, Kentaro. Sur la connexion de Weyl-Hlavatý et ses applications à la géométrie conforme. Proc. Phys.-Math. Soc. Japan (3) **22**, 595–621 (1940). [MF 2937]

Starting with G_{ij}, the relative tensor of weight $-2/n$ used by T. Y. Thomas in his development of conformal geometry, the author derives a conformal connection which is a Weyl connection and by an elimination process arrives at the conformally invariant Weyl curvature tensor. Using G_{ij} to determine arc length s (relative scalar of weight $-1/n$), he finds invariant equations analogous to the Frenet equations which introduce a sequence of conformal curvatures. A projective parameter $t(s)$, an absolute scalar, is defined and used to write the formula for the first conformal curvature vector. The vanishing of this vector characterizes the generalized circles. *J. L. Vanderslice* (Bethlehem, Pa.).

Yano, Kentaro et Mutô, Yosio. Sur la théorie des hypersurfaces dans un espace à connexion conforme. Proc. Imp. Acad. Tokyo **16**, 266–273 (1940). [MF 2944]

Using polyspherical coordinates in the tangent spaces the authors establish a generalized conformal geometry in the manner of Cartan and Veblen. They induce a conformal connection in a hypersurface by means of a normal hypersphere at each point and derive the equations of Gauss and Codazzi. Several theorems connecting generalized circles, umbilical points and autoconcurrent curves are stated and left to be proved in a later paper. *J. L. Vanderslice.*

Sasaki, Shigeo. Geometry of the conformal connexion. Sci. Rep. Tôhoku Imp. Univ., Ser. 1. **29**, 219–267 (1940). [MF 3176]

This paper deals with the conformal geometry of Riemann spaces. It is local and exceedingly formal in character. According to the author his work is presented from the unified standpoint of the theories of E. Cartan and T. Y. Thomas. The reviewer fails to see satisfactory evidence of such unification. We content ourselves with an enumeration of the main headings: I. General considerations, II. Curves in C_n, III. Fundamental quantities and equations for C_m in C_n, IV. Geometrical properties of C_{n-1} in C_n. In spite of the length of the paper the reviewer could find nothing that seemed to stand out as a mathematical contribution. *T. Y. Thomas* (Los Angeles, Calif.).

Sasaki, Shigeo. On a remarkable property of umbilical hypersurfaces in the geometry of the normal conformal connexion. Sci. Rep. Tôhoku Imp. Univ., Ser. 1. **29**, 412–422 (1940). [MF 3469]

In the author's previous study of the hypersurfaces of a space with a normal conformal connection [cf. the preceding review], two (induced and intrinsic) conformal connections for the hypersurface and a relative scalar λ defined on the hypersurface play an important role. In the present paper, any Riemann space V_n is regarded as a space endowed with a normal conformal connection and those hypersurfaces of V_n for which $\lambda\equiv 0$, as well as those for which the induced connection and the intrinsic connection coincide are investigated. Using the results of his previous paper, the author finds (1) necessary and sufficient conditions that the induced and intrinsic conformal connections of a hypersurface V_{n-1} of V_n shall coincide. A necessary condition is that $\lambda\equiv 0$ on V_{n-1}. (2) If $\lambda\equiv 0$ on a V_{n-1} of V_n, it is necessary and sufficient that V_{n-1} be a totally umbilical hypersurface. (3) In any Einstein space, the umbilical hypersurfaces whose intrinsic and induced conformal connections coincide are also Einstein spaces. (4) In a conformally flat space, the induced and intrinsic conformal connections of every umbilical hypersurface coincide. In the present work of the reviewer on the conformal differential geometry of the subspaces of a Riemann space [abstracts in Bull. Amer. Math. Soc. **46**, no. 11 (1940) and **47**, no. 3 (1941)], a scalar analo-

gous to λ is discovered which vanishes on a subspace if and only if the subspace is totally umbilical. This is analogous to the theorem stated in (2) and indicates that (2) may probably be generalized to apply to any subspace of V_n.
A. Fialkow (Brooklyn, N. Y.).

Mutô, Yosio. On some properties of subspaces in a conformally connected manifold. Proc. Phys.-Math. Soc. Japan (3) **22**, 621–636 (1940). [MF 2938]

This is a sequel to papers by the author on hypersurfaces in conformal space. The treatment of m-spaces in a conformally connected n-space is effected by the use of a family of $n-m$ orthogonal hyperspheres at each point of the subspace. This family replaces the normals of Riemannian subspace theory. The conformal displacement induced in the subspace is defined geometrically and then given analytic form. The equations of Gauss and Codazzi are found. The autoconcurrent curves of Haimovici are generalized to the present case and several theorems given relative to autoconcurrent curves, generalized circles and umbilical points. *J. L. Vanderslice* (Bethlehem, Pa.).

Takasu, Tsurusaburo. Gemeinsame Behandlungsweise der elliptischen konformen, hyperbolischen konformen und parabolischen konformen Differentialgeometrien. Proc. Imp. Acad. Tokyo **16**, 333–340 (1940). [MF 3473]

Takasu, Tsurusaburo. Gemeinsame Behandlungsweise der elliptischen Lieschen, hyperbolischen Lieschen und parabolischen Lieschen Differentialgeometrien. Proc. Imp. Acad. Tokyo **16**, 341–345 (1940). [MF 3474]

Takasu, Tsurusaburo. Gemeinsame Behandlungsweise der elliptischen Laguerreschen, hyperbolischen Laguerreschen und parabolischen Laguerreschen Differentialgeometrien. Proc. Imp. Acad. Tokyo **16**, 346–349 (1940). [MF 3475]

Announcement without proofs of various generalizations of Euclidean and non-Euclidean geometries. The announcements are primarily given in the form of catalogues of hypercomplex numbers, groups and fundamental invariants associated with such geometries. It is hoped that the foundations of these geometries will be treated in the projected third volume of the author's "Differentialgeometrien in den Kugelräumen" [cf. these Rev. **1**, 86]. *A. D. Michal.*

Wong, Yung-Chow. On two linear vector spaces associated with a vector in an L_n. Proc. Edinburgh Math. Soc. (2) **6**, 172–175 (1940). [MF 2851]

Let v_λ be a covector field in an affine manifold L_n. Then the vectors i^κ, j^κ, which satisfy the equations $i^\kappa\nabla_\kappa v_\lambda=0$, $j^\lambda\nabla_\kappa v_\lambda=0$, $\kappa, \lambda=1, 2, \cdots, n$, span a local E_{n-r} and $\bar{E}_{n-r}$, respectively, at every point of the L_n, where r is the rank of $\nabla_\kappa v_\lambda$. In a similar way we can define two local spaces $'E_{n-r'}$, $'\bar{E}_{n-r'}$ for the vector field $'v_\lambda=\sigma v_\lambda$, when σ is a scalar and r' is the rank of $\nabla_\kappa {}'v_\lambda$. The paper shows that the relation between the two pairs of spaces is completely determined by the ranks r, r_1, r_2, r_3 of the matrices

$$M=\|\nabla_\kappa v_\lambda\|,\quad M_1=\left\|\begin{matrix}\nabla_\kappa v_\lambda\\ \nabla_\kappa \log\sigma\end{matrix}\right\|,\quad M_2=\|\nabla_\kappa v_\lambda, v_\lambda\|,$$

$$M_3=\left\|\begin{matrix}\nabla_\kappa v_\lambda, & -v_\lambda\\ \nabla_\kappa \log\sigma, & 1\end{matrix}\right\|.$$

Reference is made to L. P. Eisenhart, Non-Riemannian Geometry, 1927, pp. 38–43. *D. J. Struik.*

Michal, A. D. and Mewborn, A. B. Abstract flat projective differential geometry. Acta Math. **72**, 259–281 (1940). [MF 2856]

This paper gives detailed proofs of the results announced in the Proc. Nat. Acad. Sci. U. S. A. **25**, 440–443 (1939); cf. these Rev. **1**, 176. *A. Kawaguchi* (Sapporo).

Kosambi, D. D. The concept of isotropy in generalized path-spaces. J. Indian Math. Soc. (N.S.) **4**, 80–88 (1940). [MF 2693]

This note attempts a generalization of the concept of isotropy to general path spaces of the second order [see Kosambi, Quart. J. Math. **6**, 1–12 (1935) for notation and terminology]. Dynamical considerations motivate a definition of quasi-isotropy $P_j{}^i=\delta_j{}^i\phi-\dot{x}^i q_j$, which is shown to reduce to ordinary isotropy in the Riemannian and affine path cases. For the general case the analogue of the Bianchi identities leads to important differential conditions on ϕ and q_i. Then complete isotropy (locally constant curvature) is defined in terms of ϕ and necessary and sufficient conditions found for isotropy of path spaces admitting a Finsler metric. The paper concludes with a discussion of a special type of immersion of a path space in a quasi-isotropic space with application to Milne's cosmology.
J. L. Vanderslice (Bethlehem, Pa.).

Hokari, Shisanji. Die intrinsike Theorie der Geometrie des Systems der partiellen Differentialgleichungen. Proc. Imp. Acad. Tokyo **16**, 326–332 (1940). [MF 3472]

By the intrinsic theory of the system

$$\frac{\partial^m x^i}{\partial u^{\alpha_1}\cdots\partial u^{\alpha_m}}+H^i_{\alpha_1\cdots\alpha_m}\left(u, x, \frac{\partial x}{\partial u}, \cdots, \frac{\partial^{m-1}x}{\partial u\cdots\partial u}\right)=0,$$

$$\alpha=1, \cdots, k;\quad i=1, \cdots, n,$$

is meant the invariant theory under general transformations of coordinates x and parameters u. The author previously has developed the case $m=3$, $k>1$ [Proc. Imp. Acad. Tokyo **16**, 104–108 (1940); these Rev. **2**, 22]; in the present paper he takes $m>3$. One important restriction is made, namely, the existence of an intrinsic nondegenerate tensor $h_{\alpha\beta}$ with respect to parameter transformations. The essential point of the paper is the determination of a covariant differentiation process for tensors with both coordinate and parameter indices, involving as usual two types of affine connection $\Gamma^i_{j\alpha}$ and $G^\beta_{\gamma\alpha}$. The author succeeds in finding functions of $h_{\alpha\beta}$ with the proper transformation law to serve as $G^\beta_{\gamma\alpha}$ and functions of $H^i_{\alpha_1\cdots\alpha_m}$ and $G^\beta_{\gamma\alpha}$ to serve as $\Gamma^i_{j\alpha}$. Then he shows that for $m=4, 5$ a tensor $h_{\alpha\beta}$ can be found in terms of $H^i_{\alpha_1\cdots\alpha_m}$. It is not necessarily nondegenerate, however. *J. L. Vanderslice* (Bethlehem, Pa.).

Morinaga, Kakutaro. Mathematical foundations of wave geometry. I. J. Sci. Hirosima Univ. Ser. A. **10**, 215–246 (1940). [MF 3385]

An algebraic generalization of wave geometry is attempted. The equations $\frac{1}{2}(\gamma_i\gamma_j+\gamma_j\gamma_i)=g_{ij}1$, which play a fundamental role in wave geometry, are replaced by $\alpha_i\alpha_j+\alpha_j\alpha_i=p_{ij}M$, where α_i and M are elements of a set of operators and p_{ij} are elements of a field. *A. H. Taub.*

Kawaguchi, Akitsugu und Hokari, Shisanji. Die Grundlegung der Geometrie der fünfdimensionalen metrischen Räume auf Grund des Begriffs des zwei-dimensionalen Flächeninhalts. Proc. Imp. Acad. Tokyo 16, 313–319 (1940). [MF 3470]

Kawaguchi, Akitsugu und Hokari, Shisanji. Die Grundlegung der Geometrie der n-dimensionalen metrischen Räume auf Grund des Begriffs des K-dimensionalen Flächeninhalts. Proc. Imp. Acad. Tokyo 16, 320–325 (1940). [MF 3471]

An integral of the form

$$\int_{(K)} L\left(x^i, \frac{\partial x^i}{\partial u^\beta}\right) du^1 du^2 \cdots du^K$$

is considered over a K-dimensional surface in an n-dimensional manifold. It is pointed out that the so-called Finsler geometry occurs when $K=1$ and a geometry considered by E. Cartan when $K=n-1$. Kawaguchi and Hokari are concerned essentially with the case $1<K<n-1$. In the first paper, however, the restriction $n=5$, $K=2$ is imposed and the question of the parallel displacement of a vector is discussed. Various formulas connected with this problem are derived.

In the second paper the restriction $n=5$, $K=2$ is removed. The integers n and K are now restricted to be relatively prime, the general aim being the same as in their previous paper. The paper is quite formal.

T. Y. Thomas (Los Angeles, Calif.).

Sugawara, Masao. A generalization of Poincaré-space. Proc. Imp. Acad. Tokyo 16, 373–377 (1940). [MF 3480]

The set of symmetrical matrices A of dimension n satisfying the relation $E-A\bar{A}>0$ is the space $\mathfrak{A}$, and A are its points (E is the identical matrix, $\bar{A}$ is the conjugate complex of A). The movements of this space are the transformations defined by

$$W=(U_1Z+U_2)(U_3Z+U_4)^{-1},$$

where the $2n$-dimensional matrix U satisfies the equations $U'JU=J$, $U'S\bar{U}=S$, with

$$J=\begin{pmatrix} O & E \\ -E & O \end{pmatrix} \quad \text{and} \quad S=\begin{pmatrix} E & O \\ O & -E \end{pmatrix}.$$

The group of these movements can be considered as a group of movements in a Finsler space, the linear element ds of which is defined as follows: If A and $B=A+dA$ are two points of $\mathfrak{A}$, and B is sufficiently near to A, we put

$$ds=\tfrac{1}{2}\log\frac{1+\rho}{1-\rho},$$

where ρ is the positive quadratic root of the greatest characteristic value of $dA(E-\bar{A}A)^{-1}d\bar{A}(E-A\bar{A})^{-1}$. The author also defines the distance in the large and proves the validity of the triangular inequality. *G. Fubini.*

MECHANICS

von Kármán, Theodore. The engineer grapples with nonlinear problems. Bull. Amer. Math. Soc. 46, 615–683 (1940). [MF 2658]

von Kármán appeals to pure mathematicians for cooperation with engineers who are struggling with a great variety of mathematical problems. He summarizes the intention of his Gibbs lecture as follows: "An attempt is made to show the application of analytical methods available for the solution of certain nonlinear problems in which the engineer is interested. Some gaps are shown and frontiers indicated beyond which the safe guidance of the mathematical analysis is for the time being lacking." After contrasting linear with nonlinear problems von Kármán states: "in most nonlinear problems physical reasoning is not sufficient or fully convincing, so that in these cases the questions of existence and uniqueness represent a real challenge to the mathematician." The whole field of nonlinear mathematical engineering problems is then discussed in a rather detailed survey.

1. Nonlinear vibrations result from differential equations of the type:

$$\ddot{x}+\omega^2 x=f(x,\dot{x}),$$

where x represents a deflection and ω the natural frequency of the system. Periodic solutions, corresponding to Poincaré's limit cycles, represent "self-excited" vibrations. In some limit cases there are sudden transitions between deflections of opposite sign, "relaxation vibrations" according to van der Pol. If a periodic force is applied to the system, the phenomenon of "subharmonic resonance" may result; it is important in radio-technique but also occurs in airplane vibrations; it is here treated by a perturbation method analogous to a procedure developed by Poincaré for celestial mechanics.

2. Nonlinear differential equations occur in the theory of elasticity when deflections are permitted to be large. Bending and buckling of thin rods ("elastica" problem) and plates are discussed. The differential equations for plates, derived by von Kármán in 1910, are

$$\Delta\Delta F=E(w_{xy}^2-w_{xx}w_{yy}), \quad C\Delta\Delta w=F_{yy}w_{xx}+F_{xx}w_{yy}-2F_{xy}w_{xy},$$

where E and C are constants. The most interesting phenomenon is to be found in the case of very thin plates ($C\to 0$). The solutions, which recently have been treated by asymptotic integration, will be constants except in narrow strips near the boundary where sudden changes occur. For curved arches and shells the presence of nonlinear terms provides a solution which is attained in reality even before the linear theory comes into play; in this case, therefore, the linear theory fails entirely to give account of the actual situation. A brief review is given of the rather recent general theories of nonlinear elasticity due to Murnaghan and Biot. Plastic deformations are characterized by quite a different type of nonlinear problems, namely, boundary value problems for hyperbolic differential equations of the type

$$(f_{xx}+f_{yy})^2-C(f_{xx}f_{yy}-f_{xy}^2)=0.$$

3. In the theory of fluid flow a variety of problems occur in which the differential equation (the potential equation) is linear while the boundary conditions are nonlinear. Many problems in this field have been thoroughly treated, for example, the flow around obstacles and through nozzles, also waves with large amplitudes; many other problems remain unsolved, for example, that of heavy jets over spillways or the meteorological problem of the progressing cold front; a main difficulty seems to be the determination of the proper singularity of the analytic function which represents the flow.

4. The flow of viscous fluids is governed by a nonlinear differential equation (essentially of the fourth order) for which exact solutions are known only in few instances. From an engineering point of view the limiting case that the viscosity approaches zero is more important; it had defied the powers of analysis until it was made accessible to methods of asymptotic integration through Prandtl's ingenious boundary layer theory. There remain, however, unanswered purely mathematical questions; for example, what is the flow pattern around a submerged body?

5. The interesting feature of the nonlinear differential equations of compressible fluids is that they are elliptic or hyperbolic depending on whether the velocity is below or above that of sound. For both cases some solutions are known. For problems, however, where the equation is elliptic in one part and hyperbolic in another part (a condition encountered, for example, with aerial bombs dropped from large heights), no methods are as yet available. The invention of such methods "would be an achievement both from a practical and mathematical point of view."

The paper is furnished with a great many instructive figures and supplemented by an extensive bibliography.

K. Friedrichs (New York, N. Y.).

Hydrodynamics, Aerodynamics

v. Koppenfels, Werner. Ebene Potentialströmung längs einer glatten Wand mit stückweise stetiger Krümmung. Luftfahrtforschung **17**, 189–195 (1940). [MF 3705]

The author first shows how a straight line can be transformed into a curve that is continuous and has a continuous slope but a discontinuous curvature at a certain point. Cases of infinite curvature are shown, and also cases where the curvature is step-wise constant. An example of the latter is provided by the transformation

$$Z=f(z)=\{z^{-1}+b\ln z+(\tfrac{1}{2}-\pi b)i+P(z)\}^{-1},$$

where $P(z)$ is a function that is regular at $z=0$. This transforms the axis of reals in the z-plane into a curve in the Z-plane that consists of two circular arcs of radii ρ_1 and ρ_2, tangent at $Z=0$, where $2\pi b=1-\rho_1/\rho_2$. In order to investigate the flow near such a singular point, use is made of the relation

$$\bar{\Omega}'^2(Z)\Omega''(Z)/|\Omega'(Z)|=dW/d\tau-iKW^2,$$

where $\Omega(Z)$ is the complex potential, $W=|\Omega'|$ = the magnitude of the velocity, K is the curvature of a streamline, and τ is the time, so that $dW/d\tau$ is the rate of change of W along the streamline. If $Z=f(z)$ and the flow in the z-plane is a stream of velocity W_0 parallel to the x-axis, this relation reduces to

$$-f''(z)/f'(z)|f'(z)|=|f'(z)|W_0^{-1}dW/d\sigma-iK$$

for $y=$const., where σ is the distance measured along the streamline. If this is evaluated for the transformation above, it is found that $dW/d\sigma$ is logarithmically infinite where K is discontinuous.

The practical importance of the result lies in the fact that the calculation of the laminar boundary layer along a wall by the usual Kármán-Pohlhausen procedure involves the values of the first derivative of the velocity of potential flow along the wall. If this derivative becomes infinitely great, the calculation breaks down. The author believes that in view of the corresponding rapid change of the pressure near the singular point, separation of the boundary layer from the surface must be expected. *W. R. Sears.*

Mohr, Ernst. Über die Kräfte und Momente, welche Singularitäten auf eine stationäre Flüssigkeitsströmung übertragen. J. Reine Angew. Math. **182**, 65–104 (1940). [MF 2818]

The paper is concerned with the problem of the determination of the forces and couples which the stationary flow of an ideal incompressible fluid exerts on rigid bodies placed in its path. The author states as his aim the simplified derivation of results of M. Lagally [Z. Angew. Math. Mech. **2**, 409 (1922)], with the help of results of A. Betz [Ing.-Arch. **3**, 465 (1922)]. *E. Reissner.*

Dean, W. R. Note on the slow motion of fluid. Proc. Cambridge Philos. Soc. **36**, 300–313 (1940). [MF 2601]

In two dimensional slow viscous fluid motion which takes account of the inertial terms, the stream function ψ satisfies the differential equation

$$\nu\nabla^4\psi=\psi_x\nabla^2\psi_y-\psi_y\nabla^2\psi_x, \tag{1}$$

where ν is the kinematic viscosity coefficient and the subscripts denote partial derivatives. The velocity components (u, v) are given by $(-\psi_y, \psi_x)$. The author considers the flow in the semi-infinite region above the boundary curve defined by

$$x=\sin^2\phi/2\sec\phi/2,\quad y=\cos^2\phi/2, \tag{2}$$

which has a cusp projection with vertical tangent at $\phi=0$ which hinders the flow. The region is conformally mapped upon the ζ plane such that the boundary (2) lies on the circle $|\zeta|=1$. A modification of the method of Musheliśvili is employed to determine a biharmonic function ψ_1 for the homogeneous case of (1) subject to the vanishing of ψ_1 and its normal derivative on the contour. A closed form of this solution is compared to that for motion past a projecting bar. By expansion methods involving a parameter κ containing ν, a second approximation $\psi=\psi_1+\kappa\psi_2$ of (1) is obtained in finite form, the constant κ tending to zero as the limiting case of slow motion. It is found that a shearing motion in the negative x direction persists at large distances from the cusp boundary and that a closed vortex exists on the down stream side of the projection. *D. L. Holl.*

Tsien, Hsue-Shen. Two-dimensional subsonic flow of compressible fluids. J. Aeronaut. Sci. **6**, 399–407 (1939). [MF 3279]

A method of approximation is developed in which the actual adiabatic relation between the pressure p and the density ρ is replaced by a relation $p_1-p=a_1^2\rho_1^3(\rho^{-1}-\rho_1^{-1})$ representing the tangent to the adiabatic in the (p, ρ^{-1}) plane at the point corresponding to the state of the gas in the undisturbed rectilinear flow. The corresponding Bernoulli equation is obtained for the velocity and it is shown that in this approximate theory the Mach number increases as the velocity increases. The method of the hodograph is next used and a passage from a solution for incompressible flow to one for compressible flow is effected with the aid of a transformation. Calculations are made for the case of flow over an elliptic cylinder and the results are compared with those of Glauert, Prandtl and Kaplan with the aid of graphs. Some remarks are made on the results that can be obtained with the aid of the lift and drag functions. Some difficulties which arise in the treatment of compressible flow round a lifting surface are pointed out. *H. Bateman.*

Nomura, Yûkiti. The forces on two parallel infinitely long plane plates placed in uniform flow. Sci. Rep. Tôhoku Imp. Univ., Ser. 1. **29**, 22–35 (1940). [MF 2676]

The velocity potential of the flow without circulation is expressed as a series of integrals involving Bessel functions. This method is similar to that used by the author in solving the problem of the flow around two parallel coaxial discs [Sci. Rep. Tôhoku Imp. Univ., Ser. 1. **28**, 304–318 (1940); these Rev. **1**, 284] and requires the value of the following integral

$$g(\mu, \nu)=\int_0^\infty \frac{e^{-a\xi}J_\mu(\xi)J_\nu(\xi)}{\xi}d\xi.$$

However, in the present case, μ, ν are positive integers whereas, in the case of circular discs, μ, ν are halves of odd integers. As in the case of discs, the plates experience a repulsive force and a moment which tends to make the plates perpendicular to the flow at infinity. The numerical calculation for the case of two plates with equal width is carried out in detail. It shows that with increasing distance between the plates, the force decreases more rapidly and the couple increases more slowly in the case of plates than in the case of discs. *H. S. Tsien* (Pasadena, Calif.).

Young, Gale. Some general theorems on the motion of incompressible viscous fluids. Bull. Math. Biophys. **2**, 145–155 (1940). [MF 3255]

In the first part of the paper some general theorems about the motion of a single incompressible viscous fluid are reviewed. Most of the discussions are concerned with the so-called slow motion in which the effect of second order terms in inertia forces are neglected. Theorems on the dissipation and the uniqueness of solution are the main interest. In the second part of the paper, most of the theorems given in the first part are extended to the case of several fluids moving through each other. Fick's law, that is, the drag force between two fluids is proportional to their relative velocities, is generally assumed. It is shown, for example, that the slow motion of one or more incompressible fluids is uniquely determined by the initial and boundary velocities and the equivalence classes of the external force fields (force fields whose difference has at each instant a potential). It is also shown that, if the sum of external force fields for the different fluids is equivalent to zero, the total stress in steady state is merely the sum of the stresses of each fluid in steady force-free motion in the absence of the other fluids and with its own boundary velocities. *H. S. Tsien.*

Young, Gale. A generalization of Cunningham's extension of Stoke's law for the force on a sphere. Bull. Math. Biophys. **2**, 105–108 (1940). [MF 2703]

By neglecting the inertia terms in the Navier-Stoke equations for the motion of an incompressible viscous fluid, it is well known that the solution can be expressed in terms of solid harmonics. The author uses this solution to show that the total x-force on a sphere depends only on the coefficient of x/r^3 in the expansion of the pressure fluid in solid harmonics about the center of the sphere as origin. Similar rules hold for the y-force and the z-force. The author then shows that, in the case of the flow between two concentric spheres, this coefficient, and thus the x-force, can be expressed in terms of certain averages of the fluid velocities over the surface of these two spheres. This law can be reduced to that of Cunningham [Proc. Roy. Soc. London, Ser. A. **83**, 357–365 (1910)] or to that of Stoke by specializing the conditions over the spheres.

There are few typographical mistakes in Eq. (3) of the present paper which can be found by comparing with Lamb's Hydrodynamics [p. 597, 6th ed., Cambridge, 1932]. *H. S. Tsien* (Pasadena, Calif.).

Nomura, Yûkiti. On the wave resistance of a plane plate. Sci. Rep. Tôhoku Imp. Univ., Ser. 1. **29**, 268–286 (1940). [MF 3177]

An infinitely long strip of plate of width $2a$ moves in an infinitely deep stream parallel to the undisturbed free water surface with a velocity c. The plate is held perpendicular to the direction of motion. The distance between the center line of the plate and the undisturbed water surface is f. The problem is to find the wave resistance and the moment acting on the plate. The author used an iteration method. He first ignored the boundary condition at the free surface, and found the velocity potential Φ_0 for flow around the plate. Then a second velocity potential Φ_1 is added, which does not satisfy the boundary condition at the plate, but satisfies the boundary condition at great depth. The potential Φ_1 is determined by the boundary condition that pressure = constant at the free water surface. The well-known expedients of neglecting second order quantities of disturbance and a small frictional force are used. The boundary condition at the surface of plate is then satisfied by the addition of a third potential Φ_2, which introduces an uncorrected small pressure disturbance at the free water surface. The resistance and the moment in the plate are calculated by integrating the pressure over the surface of the plate. The resistance R is found to agree with H. Lamb's formula $R=\frac{1}{2}\rho g A^2$, where A is the amplitude of the regular waves in the rear of the plate, ρ is fluid density and g the gravitational acceleration. Numerical results are given for various values of f and $c/\sqrt{gf}$.

The second problem of a circular disc of radius a moving in the direction of its axis below the water surface is treated in a similar way. The resistance is found to agree with the calculations of T. H. Havelock [Proc. Roy. Soc. London (A) **131**, 275–285 (1931)]. Numerical results are given for the case of $f=5a$. *H. S. Tsien* (Pasadena, Calif.).

Ringleb, Friedrich. Exakte Lösungen der Differentialgleichungen einer adiabatischen Gasströmung. Z. Angew. Math. Mech. **20**, 185–198 (1940). [MF 3111]

If w denotes the velocity in two dimensional flow and θ its direction, the equations connecting the velocity potential Φ and the stream function Ψ in steady motion may be written in the form

$$\rho w c^2\Phi_w=-\rho_0(c^2-w^2)\Psi_\theta, \quad \rho\Phi_\theta=\rho_0 w\Psi_w,$$

and in the adiabatic case the equation for Ψ is

$$w^2(1-\alpha w^2)\Psi_{ww}+w(1-\beta w^2)\Psi_w+(1-\gamma w^2)\Psi_{\theta\theta}=0,$$

where $2c_0^2\alpha=k-1$, $2c_0^2\beta=k-3$, $2c_0^2\gamma=k+1$. When Ψ depends only on w a solution is

$$\Psi=C[W+\tfrac{1}{3}W^3+\tfrac{1}{5}W^5-Q_0(W)]+C',$$

where $W=(1-\alpha w^2)^{\frac{1}{2}}$. The stream-lines for this type of vortex motion are drawn by using the relations

$$\rho wx=\rho C_1\sin\theta+\rho_0 C_2\cos\theta,\ \rho wy=\rho_0 C_2\sin\theta-\rho w C_1\cos\theta,$$
$$\Psi=C_1[W+\tfrac{1}{3}W^3+\tfrac{1}{5}W^5-Q_0(W)]+C_2\theta,$$
$$2Q_0(W)=\log(1+W)-\log(1-W),$$

the solution for a source being added. The phenomenon of flow shock now appears. Flow round an edge is discussed by seeking a solution of type $\Psi=P(w)Q(\theta)$ and figures are

drawn to indicate the difference between a compressible fluid and an incompressible fluid in this case. A general method of integration is sketched and applied to the double source. *H. Bateman* (Pasadena, Calif.).

Stümke, H. Rotationssymmetrische Gleichgewichtsstörungen in einer isothermen Atmosphäre nebst einem Modellversuch mit rotierender Flüssigkeit. Z. Geophys. **16**, 127–149 (1940). [MF 2765]

In a paper presented at the International Meteorological Conference, Edinburgh, 1936, L. Prandtl represented tentatively cyclonal and anticyclonal movements by the streamfield caused by a negative and a positive "source," respectively, in an isothermically stratified atmosphere. While Prandtl simplified the problem by considering a two dimensional case, the present study gives the discussion for the axial-symmetric case, thus coming somewhat closer to the real aerological conditions. The problem leads to a boundary value problem of the equation:

$$\frac{\partial^2\psi}{\partial X^2}+\frac{\partial^2\psi}{\partial Z^2}+\frac{1}{X}\frac{\partial\psi}{\partial X}+\frac{\partial\psi}{\partial Z}=0,$$

X and Z being cylindric coordinates made dimensionless. In an appendix the investigation is carried out for the case of an incompressible liquid and the results checked by an experiment. *P. Nemenyi* (Iowa City, Iowa).

Görtler, H. Über den Einfluss der Wandkrümmung auf die Entstehung der Turbulenz. Z. Angew. Math. Mech. **20**, 138–147 (1940). [MF 3081]

A laminar two-dimensional flow along the circumference of a circular cylinder of radius R is first assumed. The velocity of this flow is $U(y)$, where y is the distance normal to the cylindrical surface; $U(0)=0$, and $U(y)=$constant$\cdot(R+y)/R$ for $|y|>\delta$, where δ is the so-called thickness of the boundary layer. The author then superimposes on this flow a two-dimensional disturbance given by the stream function

$$\psi(x, y, t)=\varphi(y)e^{i(\alpha x-\beta t)}=\varphi(y)e^{i\alpha(x-ct)},$$

where x is the distance along the circumference of the cylindrical surface and t is the time; α is taken to be real. The problem is to investigate the stability of this disturbance by means of hydrodynamical equations, assuming that the amplitude of the disturbance is small compared to $U(y)$ and only linear terms of ψ are retained in the equations. Given $U(y)$ and α, and the boundary conditions at $y=0$ and $y=\delta$, the problem reduces to that of finding the eigenvalue of β. If the imaginary part of β is greater than zero, the motion is unstable. The author investigated the case of large values of Reynold's number $U(\delta)\delta/\nu$, where ν is the kinematical viscosity. The method used is that of W. Tollmien [Nachr. Ges. Wiss. Göttingen. Fachgruppe 1. (N.F.) **1**, 79–114 (1935)] who neglected the effect of viscosity except at the neighborhood of the point where $U(y)=c$. It is shown that if $U''+U'/R$ changes sign during the interval $0\leqq|y|\leqq\delta$ there is a value of $\alpha>0$ which gives unstable motion. Therefore a concave surface has stabilizing effect and a convex surface has de-stabilizing effect compared with a plane surface. *H. S. Tsien* (Pasadena, Calif.).

Maue, A. W. Zur Stabilität der Kármánschen Wirbelstrasse. Z. Angew. Math. Mech. **20**, 129–137 (1940). [MF 3080]

The author considers the problem of the stability of a double row of vortices, which was first treated by von Kármán [Nachr. Ges. Wiss. Göttingen **12**, 509 (1911); **13**, 547 (1912); H. Lamb, Hydrodynamics, 6th ed., Cambridge, 1932, pp. 225–229]. In the new treatment function-theoretical methods are applied, and closed expressions are obtained in place of the infinite sums of the earlier works. Arbitrary first-order disturbances are assumed, and the following stability condition is obtained:

$$\sinh(h\pi/l)=\sin(d\pi/l), \tag{1}$$

where h is the distance between the rows, l is the spacing of the vortices in each row and d is the distance between corresponding vortices of the two rows, measured parallel to the rows. The cases considered by von Kármán were $d=0$ (symmetrical) and $d=l/2$ (staggered); he concluded that the former is unstable for all finite, non-vanishing values of h and that the latter is stable provided that $\sinh(h\pi/l)=1$. These results are now obtained as special cases according to Eq. (1). In von Kármán's case the motion of the vortices is parallel to the rows; in all other cases represented by Eq. (1) they move in oblique directions, assuming the fluid to be at rest at infinity. Such arrangements may be produced by flow past unsymmetrical or inclined bodies, and will be accompanied by "lift" as well as "drag" on the bodies. These force components can be calculated by the methods used by von Kármán.

W. R. Sears (Pasadena, Calif.).

Konstantinov, V. A. Über helikoidale Wirbel in Anwendung zur aerodynamischen Berechnung der Luftschrauben und Windkraftmaschinen. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 688–693 (1940). [MF 3614]

Some general theorems concerning the induced velocities due to the trailing helicoidal vortex surface from a propeller are obtained by means of the following relations:

$$v_x(\bar{z})=-\frac{1}{4\pi R}\int_0^1\frac{d\Gamma}{d\bar{r}}d\bar{r}\sum_{k=1}^{i}X_k(\bar{r},\bar{z}),$$

$$v_y(\bar{z})=-\frac{1}{4\pi R}\int_0^1\frac{d\Gamma}{d\bar{r}}d\bar{r}\sum_{k=1}^{i}Y_k(\bar{r},\bar{z}),$$

$$v_z(\bar{z})=-\frac{1}{4\pi R}\int_0^1\frac{d\Gamma}{d\bar{r}}d\bar{r}\sum_{k=1}^{i}Z_k(\bar{r},\bar{z}).$$

At a distance z from the axis, v_x, v_y and v_z are the induced velocities at the blade in the axial direction of the helicoidal surface, in the circumferential direction and in the radial direction, respectively. R is the radius of the propeller, Γ the circulation around the blade at r, i the number of equally spaced blades, and $\bar{z}=z/R$, $\bar{r}=r/R$. The quantities X_k, Y_k and Z_k are the "influence coefficients" calculated by infinite integrals extending downstream along the helical vortex filaments. It is first shown that v_x and v_y at the blades and halfway between the blades are exactly half of the corresponding quantities in a section far downstream. The corresponding value of v_z in the downstream section is, however, zero.

By direct integration of the sum $X_k\lambda_0+Y_k\bar{z}$, where λ_0 is the actual pitch of the propeller, and an examination on the character of flow around the helicoidal vortex surface, the author obtains two relations between $\sum_{k=1}^{i}X_k$, $\sum_{k=1}^{i}Y_k$, λ_0, $\bar{z}$ and the "induction factor" calculated by T. Moriya [J. Fac. Engr. Tokyo Imp. Univ. **20**, 147 (1933); J. Soc. Aeronaut. Sci. Nippon **3**, 12 (1936)]. These relations can be used to determine the two unknowns $\sum_{k=1}^{i}X_k$ and $\sum_{k=1}^{i}Y_k$, and thus v_x and v_y which are needed for aerodynamic calculations of

the propeller. Another procedure is suggested by the author: $\sum_{k=1}^{i} X_k$ is first calculated. Then by using one of the relations just mentioned between $\sum_{k=1}^{i} X_k$ and $\sum_{k=1}^{i} Y_k$, $\sum_{k=1}^{i} Y_k$ can be determined. By a certain transformation of the integral for X_k, it is found that X_k can be expressed as a sum of two terms. The first term will give a value of v_z which corresponds to the well-known result obtained by assuming $i=\infty$. The second term is thus a correction for finite number of blades. This correction term can be calculated to the desired accuracy by a series. The paper concludes with a general discussion on the absolute magnitudes of $\sum_{k=1}^{i} X_k$ and $\sum_{k=1}^{i} Y_k$ which for the case i finite are found to be always larger than the corresponding quantities for the case $i=\infty$. *H. S. Tsien* (Pasadena, Calif.).

Schmieden, C. Über Tragflügelströmungen mit Wirbelablösung. Luftfahrtforschung 17, 37–46 (1940). [MF 3704]

In the usual theory of two-dimensional airfoils the so-called Kutta-Joukowski condition of finite velocity at the trailing edge requires (for non-vanishing trailing-edge angles) a stagnation point there, and the flow is required to leave the trailing edge parallel to the bisector of the trailing-edge angle. This is not observed in experiments; instead, the flow separates from the upper (suction) side near the trailing edge and flows off the lower side tangentially at the edge, leaving a vortical "wake." The author replaces this wake by two vortex sheets of equal and opposite constant strengths, with stationary fluid between them. One sheet leaves the lower surface tangentially at the trailing edge; the other leaves the upper surface at a "separation point" to be determined. This problem is handled in a manner similar to the usual Helmholtz discontinuous potential problem, using the transformation $\zeta=-\log q+i\theta$, where $qe^{i\theta}=dw/dz$ and $w(z)$ is the complex potential function in the airfoil plane. It is shown that for a given profile at a given attitude there is a single infinity of solutions with various points of separation (including as one limiting case the Helmholtz solution) which involve infinitely long wakes. Of these there is a unique solution for which the wake breadth decreases asymptotically toward zero, giving a finite dead-water region. There is also a second infinity of solutions in which the wake has a finite length, including as a limiting case the ordinary Kutta-Joukowski case.

The unique case mentioned is treated in detail for a flat-plate airfoil and for a family of thin symmetrical profiles. By a second transformation the ζ-plane is mapped on a τ-plane where the airfoil becomes a semi-circular arc and the free streamlines become its bounding radii. The force on the airfoil is shown to be a pure lift (perpendicular to the undisturbed velocity). For a flat plate the lift is smaller than the Kutta-Joukowski value; the curve of lift against angle is a parabola tangent to the Kutta-Joukowski straight line at the origin. The pressure distribution for a flat plate at 12° incidence is shown in comparison with the usual distribution. The author deduces that his conclusions must hold for all reasonably thin profiles, such as are used in practice. A method is given by which the separation point of the upper vortex sheet might be calculated approximately for any given profile. *W. R. Sears* (Pasadena, Calif.).

Vandrey, F. Beitrag zur Theorie des Tragflügels in schwach inhomogener Parallelströmung. Z. Angew. Math. Mech. 20, 148–152 (1940). [MF 3082]

It is known that the aerodynamic theory of the aerofoil of finite span l in a homogeneous parallel stream of velocity V leads to the problem of determining a potential $\phi(x, y)$ regular outside the slit $y=0$, $-l/2<x<l/2$, odd in y and satisfying along the slit the condition

$$L(\phi)\equiv\left(\frac{\partial\phi}{\partial y}\right)_{y=0}+k[\phi(x, y+0)-\phi(x, y-0)]=\alpha(x)V.$$

If V is not uniform but piecewise constant, say $V=V_m$ in a region B_m, then the problem consists in finding potentials ϕ_m satisfying the following conditions: Along the common boundary of B_m and B_p, $V_m\phi_m=V_p\phi_p$, $V_p\partial\phi_m/\partial n=V_m\partial\phi_p/\partial n$, and along the part of the slit in B_m, $L(\phi_m)=\alpha V_m$. The author shows that, when $V_m=(1+\epsilon_m)V$, then in first approximation $\phi_m=(1-\epsilon_m)\phi$ and $L(\phi)=(1+2\epsilon_m)\alpha V$, where now ϵ_m may be a continuous function of x. This indicates that an aerofoil in slightly inhomogeneous parallel flow behaves like an aerofoil in homogeneous flow with a slightly changed spanwise variation of angle of attack. *E. Reissner.*

Achyeser, N. Über die Konstruktion des Stromes um ein dünnes Profil. Acad. Sci. RSS Ukraine. Rec. Trav. [Zbirnik Prace] Inst. Math. 1940, no. 4, 151–156 (1940). (Ukrainian. Russian and German summaries) [MF 3233]

The force and couple acting on a two-dimensional thin airfoil in a steady stream and the velocities at its surface are calculated for cases in which the profile deviates slightly from a circular arc. The formulae obtained are analogous to those given by M. Munk [Rep. 142, U. S. Nat. Advis. Comm. for Aero. (1922); also W. F. Durand, Aerodynamic Theory, vol. II, § E II, 9, pp. 43–48] for airfoil profiles that deviate slightly from a straight line. *W. R. Sears.*

Dickmann, H. E. Grundlagen zur Theorie ringförmiger Tragflügel (frei umströmte Düsen). Ing.-Arch. 11, 36–52 (1940). [MF 3425]

The problem treated is that of a ring-shaped wing, or a short nozzle of airfoil cross section immersed in a fluid. Experience has shown that the effectiveness of a propeller is increased if it is placed inside such a ring. The ring is assumed to have axial symmetry; it is under the influence of a uniform axial flow and a steady interference flow (for example, from a propeller) which is axially symmetrical but otherwise arbitrary. The ring is replaced by a ring-shaped vortex sheet, and the usual approximations of the thin-airfoil theory are made regarding the profile and the conditions at its trailing edge. The circulation distribution $\gamma(x)$ over the chord is then given by a nonlinear integral equation of the first kind, in which the kernel involves elliptic integrals. This kernel is expressed in series form, convergent for $\delta>2$, where $\delta=4R/l$, $R=$radius of ring, $l=$chord length of profile. Finally the profile curve $y(x)$ and the interference velocities $F_x(x)$, $F_y(x)$ are expressed in Fourier series, and after application of the boundary condition at the surface of the profile a system of linear equations is obtained for the Fourier constants of $\gamma(x)$. For $R\to\infty$ the solution approaches that for the two-dimensional thin airfoil. For $l\to 0$ it becomes the case of the vortex ring. The resultant axial force on the ring is calculated; it disappears if there is no radial interference component $F_y(x)$, since the ring produces no "trailing vortices." If $F_y(x)=\text{const.}=e_1$, a thrust is produced equal to $2\pi R\rho\Gamma e_1$, where Γ is the total circulation about the profile and ρ is the fluid density.

As an example, the case of a cylindrical tube is treated in detail. For $\delta=5$ it is found that four terms suffice in the

Fourier developments. For this case, assuming $F_y(x)=\text{const.}$, $\gamma(x)$ and the induced axial velocity $c(x)$ are plotted; Γ is 12 percent greater than for a plane airfoil at the same angle of attack. Also Γ is plotted against δ for $3\leqq\delta\leqq 6$.

W. R. Sears (Pasadena, Calif.).

von Mises, Richard. Mathematical problems in aviation. Amer. Math. Monthly **47**, 673–685 (1940). [MF 3645]

Address delivered at the meeting of the Mathematical Association of America on September 9, 1940.

* **Hayes, Thomas J. Exterior Ballistics. A reprint of Chapter X, Exterior Ballistics, and Chapter XII, Bombing from Airplanes, from Elements of Ordnance.** John Wiley & Sons, Inc., New York, 1938. 98 pp. $1.00.

Bucerius, H. Neubegründung der äusseren Ballistik. Astr. Nachr. **270**, 66–73 (1940). [MF 2680]

The trajectory of a projectile is the solution of the differential system

$$\frac{d^2x}{dt^2}=-c\frac{f(v)}{v}\frac{dx}{dt},\quad \frac{d^2z}{dt^2}=-c\frac{f(v)}{v}\frac{dz}{dt}-g,$$

with the conditions $x=z=0$ when $t=0$, $x=X$, $z=0$ when $t=T$. Regarding this as a non-linear boundary value problem, the author replaces it by the system of integral equations

$$x(t)=\frac{X}{T}t+\int_0^T G_1(t,t')\frac{cf(v)}{v}\frac{dx}{dt}dt',$$

$$z(t)=\int_0^T G_1(t,t')\left\{\frac{cf(v)}{v}\frac{dz}{dt}+g\right\}dt',$$

in which Green's function $G_1(t,t')$ is defined by

$$G(t,t')=\begin{cases} t(1-(t'/T)), & t\leqq t',\\ t'(1-(t/T)), & t\geqq t'.\end{cases}$$

The solution for the case of no resistance (parabolic path) is first obtained. Then the case $f(v)=v^2$ is attacked by undetermined coefficients [cf. A. Hammerstein, Acta Math. **54**, 117–176 (1930)] using

$$x(t)=t+\sum_1^\infty a_\nu\sin\nu\pi t,\quad z(t)=\sum_1^\infty c_\nu\sin\nu\pi t$$

(where units have been changed to make $X=T=1$). To illustrate the procedure the author determines the first two or three coefficients by successive approximations (ignoring the rest of the series) and applies the method to a typical example. The results are admittedly not accurate enough for modern ballistic requirements, so that more terms would have to be used. The integral equations above are then modified so as to apply to bombs dropped from airplanes, including the case of dive-bombers. The procedure to be followed when $f(v)$ is obtained from tables rather than from an analytical expression is not indicated. *W. E. Milne.*

García, Godofredo. Differential equations of the movement of the center of gravity of a projectile in the space, taking into account the forces acting on the movement around the center of gravity. Deviation of the projectile. Actas Acad. Ci. Lima **3**, 65–69 (1940). (Spanish) [MF 3265]

Theory of Elasticity

Prager, W. On an analogy between the fundamental equations of hydrodynamics and elastostatics. Rev. Fac. Sci. Univ. Istanbul (A) **5**, 41–43 (1940). (English. Turkish summary) [MF 3241]

The problems in question are the plane irrotational flow of an ideal fluid and the plane stress problem. The analogy is a purely formal one. The author, on the basis of this analogy, is able to give, for example, a definition of conjugate biharmonic functions analogous to the definition of conjugate harmonic functions. *J. J. Stoker.*

Deuker, Ernst-August. Die Grundgleichungen der klassischen Elastizitätstheorie in allgemeinen Koordinaten. Deutsche Math. **5**, 94–107 (1940). [MF 2811]

A derivation of the fundamental equations of the linear theory of elasticity with respect to general curvilinear (not necessarily orthogonal) coordinate systems. The notation of tensor calculus is used. One interesting novelty is that it appears inconvenient to work with the quantities stress (force per unit area) and strain (a dimensionless quantity) but rather with more general quantities. *J. J. Stoker.*

Locatelli, P. Sul principio di Menabrea. Boll. Un. Mat. Ital. (2) **2**, 342–347 (1940). [MF 2987]

The principle of Menabrea states that the elastic energy of a body in perfect elastic equilibrium is a minimum with respect to any possible system of stress-variation compatible with the equations of the statics of continua (as well as with the boundary conditions). Using the tensors introduced by Finzi [Atti Accad. Naz. Lincei. Rend. **19**, 578 (1934)], the author transforms this theorem so that it can be stated as a minimum theorem for arbitrary variations. The author shows the relation of his theorem with the 6 compatibility equations of de Saint Venant.

P. Nemenyi (Iowa City, Iowa).

Kiltchevsky, N. A. Les équations fondamentales de l'équilibre des enveloppes élastiques et quelques méthodes de leurs intégration. I and II. Acad. Sci. RSS Ukraine. Rec. Trav. [Zbirnik Prace] Inst. Math. **1940**, no. 4, 83–149; no. 5, 73–97 (1940). (Ukrainian. Russian and French summaries) [MF 3232, 3751]

The paper contains a derivation of the basic equations of the theory of shells in an invariant form. The stress tensor is replaced by a system of forces and moments acting on the edges of the middle surface of the shell, and all quantities characterizing the stressed and strained states are expressed through the displacements of the middle surface and their derivatives with respect to the intrinsic coordinates of the middle surface. Account is taken of the effect of surface loads on the behavior of the normal to the middle surface of the shell. It is supposed that the above-mentioned generalization of Kirchhoff's hypothesis may prove significant in the study of the stability of shells and will account, in part, for the discrepancies between observed and theoretical results. The author compares his equations with those obtained by W. Flügge [Ing.-Arch. **3**, 463–506 (1932)] and by K. v. Sanden and F. Toelke [Ing.-Arch. **3**, 24–66 (1932)], and is led to the conclusion that these investigators have neglected some terms of the same order of magnitude as those appearing in their final equations. The author's equations specialize to the known particular systems of elastostatic equations. *I. S. Sokolnikoff.*

Knol, D. K. On singular points of a two-dimensional field of stresses. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **8**, 84–100 (1939). (Russian) [MF 3341]

The isoclinic lines in the plane field of stress are the lines of constant argument of the quantity

$$\varphi \equiv 2X_y + i(X_x - Y_y).$$

The paper is concerned with a study of the character and properties of the neighborhoods of essential singular points of isoclinic lines by investigating the points for which $|\varphi| = 0$. *I. S. Sokolnikoff* (Madison, Wis.).

Westergaard, H. M. Effects of a change of Poisson's ratio analyzed by twinned gradients. J. Appl. Mech. **7**, A-113–A-116 (1940). [MF 2685]

The author deduces the following theorem. If Θ' represents the "bulk stress" for the solution of the fundamental equation of elasticity for a given set of boundary conditions of an elastic solid with Poisson's ratio equal to m, for the same set of boundary conditions but a different Poisson's ratio (μ) the solution can be secured by adding to the original solution a state of displacements ϱ'' which is the "twinned gradient" of a scalar ϕ:

$$2G\varrho'' = -\mathbf{i}\frac{\partial\phi}{\partial x} - \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z},$$

where ϱ'' is the displacement vector, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ the unit vectors in the directions of the three axes of coordinates, and G the modulus of elasticity in shear, satisfying the equation

$$\frac{\mu}{1-\mu}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{\partial^2\phi}{\partial z^2} = \frac{m-\mu}{(1+m)(1-\mu)}\cdot\Theta'.$$

The stress system corresponding to ϱ'' is expressed in terms of ϕ and it is shown that to maintain equilibrium a system of mass forces

$$\mathbf{K}'' = \mathbf{i}\frac{\partial\Delta\phi}{\partial x} + \mathbf{j}\frac{\partial\Delta\phi}{\partial y},$$

where Δ is the Laplace operator, is necessary. The author shows how this theorem leads immediately to the solution of such three-dimensional problems of elasticity as Boussinesq's and Carrut's problem of the semi-infinite solid (the extremely simple solutions for $m = \frac{1}{4}$ serve as starting point). Similarly the deductions leading to the solution of a rotating thick disk are shortened by the new analytic device. *P. Nemenyi* (Iowa City, Iowa).

Reissner, Eric. A contribution to the theory of elasticity of non-isotropic materials (with applications to problems of bending and torsion). Philos. Mag. (7) **30**, 418–427 (1940). [MF 3693]

The author investigates the solution of the two-dimensional problem of equilibrium of an anisotropic solid for which the elastic moduli E_y and ν_y, associated with the y-axis, vanish. An application is made to (a) bending of a narrow rectangular beam, (b) bending of a wide-flanged box beam, (c) torsion of a box beam. The resulting formulas are intended to apply to those materials for which the ratio of the moduli E_y/E_x is negligibly small.

I. S. Sokolnikoff (Madison, Wis.).

Einaudi, Renato. Un problema fondamentale della dinamica dei sistemi elastici. Ann. Mat. Pura Appl. (4) **19**, 1–33 (1940). [MF 3042]

Let C be an elastic body, F its boundary and t the time. The author considers the displacement vector $S_i(P, t)$ of a point P of C under the influence of the force vector $X_i(P, t)$ $(i = 1, 2, 3)$, that is, he deals with the following mixed problem for S_i:

$$(\lambda+\mu)\frac{\partial}{\partial x_i}\sum_{j=1}^{3}\frac{\partial S_j}{\partial x_j} + \mu\Delta S_i + \rho X_i = \rho\frac{\partial^2 S_i}{\partial t^2},$$

$$(*) \qquad S_i(P, 0) = s_i(P), \quad \left(\frac{\partial S_i}{\partial t}\right)_{t=0} = v_i(P),$$

$$\lim_{P\to M} S_i(P, t) = \sigma_i(M, t), \quad M \subset F;\ i = 1, 2, 3,$$

where s_i, v_i and σ_i are given functions and λ, μ, ρ constants. Besides conditions concerning the differentiability of these functions and the geometric nature of the boundary F the author assumes:

(i) $$s_i(M) = \sigma_i(M, 0), \quad v_i(M) = \left(\frac{\partial\sigma_i}{\partial t}\right)_{t=0}, \qquad M \subset F;$$

(ii) equation (*) still holds for $t = 0$ if on its left side the vector S_i is replaced by the vector s_i, and by σ_i on its right side; (iii) the equation obtained from (*) by differentiating with respect to t still holds for $t = 0$ if on its left side the vector $\partial S_i/\partial t$ is replaced by the vector v_i, and by $\partial\sigma_i/\partial t$ on its right side. Under these hypotheses existence and uniqueness proofs are given. The methods are similar to those used by the author [Il problema misto per l'equazione delle onde, Atti Accad. Sci. Torino **74**, 470–480 (1939); these Rev. **1**, 316]. *E. Rothe* (Oskaloosa, Iowa).

Rosenblatt, Alfred. Sur les ondes de gravité. Actas Acad. Ci. Lima **3**, 37–42 (1940). [MF 2710]

Rosenblatt, Alfred. Sur les ondes de gravité. Revista Ci., Lima **42**, 479–515 (1940). [MF2902]

The formula of Cauchy-Poisson for the waves of emersion produced in an infinitely deep, incompressible liquid [see Lamb, Hydrodynamics, fifth ed., p. 361, formula 1] is derived without the customary assumptions about the potential or the shape of the free surface. Expanding the potential as a power series in terms of a parameter, the author shows that the first term is the Cauchy-Poisson approximation and derives integral equations by means of which the higher terms may be calculated.

B. Friedman (Chicago, Ill.).

Rosenblatt, Alfred. Sur la propagation des ondes de Rayleigh dans les milieux transversalement isotropes (milieux de Rudzki). Actas Acad. Ci. Lima **3**, 70–74 (1940). [MF 3266]

Rosenblatt, Alfred. Sur la propagation des ondes sismiques dans les milieux transversalement isotropes. Note II. Actas Acad. Ci. Lima **3**, 91–94 (1940). [MF 3268]

Föppl, O. Graphische Berechnung von Eigenschwingungszahlen. Ing.-Arch. **11**, 178–191 (1940). [MF 2808]

The vibration frequencies of a rotating crankshaft are determined by a graphical method of successive approximation. An analogue is utilized between the given shaft which is assumed to carry rotating masses at various points and a vibrating elastic string which is loaded with masses at corresponding points. In this analogy, however, the

rotating masses and distances between masses are interchanged, so that they correspond respectively to distances between masses and masses. The fundamental vibration frequency for the string can be expressed in terms of the displacements of the masses on the string. These displacements can be found by repeated approximation, using the condition that in the extreme position the string is in static equilibrium under the inertial forces of the masses. The method can be extended to second order frequencies. A numerical example is worked out for the case of a shaft carrying a six cylinder engine, dynamo and water pump.

The elastic curve and vibration frequencies of a loaded beam which is vibrating are considered in the second part of the paper. The elastic curve, which differs from that of static beam under the same loading, is equivalent to that of an elastic string with a continuous loading whose amount at any point is proportional to the ordinate of a certain bending moment diagram. This bending moment diagram is constructed not from the given loads but from those loads times the displacements at their points of application; but these displacements are unknown. A method of successive approximation is therefore used, starting with an assumed elastic curve. The frequencies can be computed from the elastic curve and the bending moment diagram. A numerical example is worked out for a shaft with two different cross sections, the diameter changing at the middle of the shaft, and two non-symmetrically placed, unequal loads.

P. W. Ketchum (Urbana, Ill.).

Colwell, R. C., Stewart, J. K. and Arnett, H. D. Symmetrical sand figures on circular plates. J. Acoust. Soc. Amer. 12, 260–265 (1940). [MF 2858]

Chladni nodal systems, as observed in many experiments, are much more varied than those obtained mathematically as zero lines of characteristic functions of the corresponding eigenvalue problem. By explicit calculation the authors show that many of the Chladni phenomena for circular plates and membranes are well represented as nodal lines of a motion consisting of the superposition by several eigenvibrations. *R. Courant* (New York, N. Y.).

Maier, Erich. Biegeschwingungen von spannungslos verwundenen Stäben, insbesondere von Luftschraubenblättern. Ing.-Arch. 11, 73–98 (1940). [MF 2681]

Methods for calculation of the natural frequencies of flexural vibration of a beam which is "twisted" when in the stress-free state are derived. The expression "twisted" means that the principal axes of inertia of the cross sections of the beam vary in direction from section to section before any forces are applied, as is the case with propeller and turbine blades. The boundary conditions considered are those of the clamped-free beam. Results of H. Reissner [Ing.-Arch. 4, 557–569 (1933)] proving that small initial curvature of the axis of the beam and coupling with torsional vibrations can be neglected are taken over. The energy and variational equations for the beam considered rotating about a line perpendicular to the beam, and the integral equations for the non-rotating case are derived. The eigenvalues and eigenfunctions for these problems are shown to have the properties one is accustomed to expect, with one notable exception: since the eigenfunctions are in general space curves, none of them need possess nodes. The case of a non-rotating beam with a constant cross section and "twist" linearly increasing with distance along the axis is treated numerically by means of the methods of Ritz and Grammel [Ing.-Arch. 10, 35 (1939)]; upper and lower bounds for the two lowest natural frequencies are obtained. The calculations indicate that the method of Grammel is far the better of the two. Calculations are also given for a special propeller blade (non-rotating case) by the method of Grammel and compared with experimental results. It is found that the two lowest frequencies as obtained by calculation check the experimentally determined ones quite accurately. One additional interesting result (among others) is that the "twist" appears not to affect the lowest natural frequency to any great extent, that is, this frequency is nearly the same as that of a beam of the same cross section without twist. The next higher frequency, however, is lowered considerably (in the case of the propeller blade mentioned above by 15 to 20 percent). *J. J. Stoker.*

Poritsky, H. and Robinson, C. S. L. Torsional vibration in geared-turbine propulsion equipment. J. Appl. Mech. 7, A-117–A-124 (1940). [MF 2686]

The mechanical system consisting of high- and low-pressure turbines, gears, shafts and propeller in a ship is replaced by an equivalent system of inertias and elastic shafts. The vibration of the system due to an oscillatory propeller torque is calculated. Viscous damping is neglected except in the propeller torque, and solid-friction damping is represented approximately by the introduction of complex numbers as shaft-stiffness constants. (The usual complex representation of oscillating quantities is employed.) The forced vibrations are calculated by the "direct method" and also by resolution into normal coordinates. The damping is supposed to be small, so that the critical frequencies are very near to the natural frequencies of the undamped system. *W. R. Sears* (Pasadena, Calif.).

Thorne, C. J. and Atanasoff, J. V. A functional method for the solution of thin plate problems applied to a square, clamped plate with a central point load. Iowa State Coll. J. Sci. 14, 333–343 (1940). [MF 3315]

The authors indicate how a number of approximation methods for solving problems associated with linear differential equations, for example, those of Ritz and Trefftz, can be subsumed under one general scheme, which they call the "functional method." The method, in one of its possible forms, is applied to the numerical solution of the problem indicated in the title. The results are in good accord with those obtained by other methods. *J. J. Stoker.*

Federhofer, K. Knickung der Kreisplatte und Kreisringplatte mit veränderlicher Dicke. Ing.-Arch. 11, 224–238 (1940). [MF 2809]

The thickness of the plate is taken to be given by the equation $h=cr^n$. The inclination of the symmetrically buckled plate satisfies a differential equation of the second order whose solution, except for the values 0 and 4/5 of the exponent n, can only be expressed in infinite series. When the solution of the differential equation is entered in the boundary conditions a complicated equation is obtained for the determination of the critical uniform pressure on the circumference of the plate. Results are given for the case $n=4/5$. In this particular case the solution of the differential equation can also be expressed in terms of Bessel's functions with imaginary arguments. This solution, however, cannot be used to advantage for lack of suitable tables. The case $n=0$ [E. Meissner, Schweiz. Bauztng. 101, 87 (1933)] leads to Bessel's function with real arguments. Results for other values of n are to be published by H. Egger.

H. W. March (Madison, Wis.).

Federhofer, K. Berechnung der Auslenkung beim Ausbeulen dünner Kreisplatten. Ing.-Arch. **11**, 118–124 (1940). [MF 2683]

The paper is concerned with an investigation of the deformations of a plane circular plate buckled by stresses distributed uniformly along the edge of the plate. The author derives a non-linear differential equation for the deflection of the buckled plate, and treats it with the aid of Galerkin's method. A formula for the deviation of the middle plane of the plate from its position of stable equilibrium is obtained. *I. S. Sokolnikoff* (Madison, Wis.).

Neuber, H. Über das Kerbproblem in der Plattentheorie. Z. Angew. Math. Mech. **20**, 199–209 (1940). [MF 3112]

The author presents a theory of the thick plate making use, as he has done in previous work [Kerbspannungslehre, Berlin, 1937], of a combination of harmonic functions to express the stress function. Applications are made to the problems of the bending of a notched plate and of a plate with an elliptical hole, under the action of couples applied at the ends of the plates. The concentrations of stress in the neighborhood of the notches and of the hole are determined. *H. W. March* (Madison, Wis.).

Zanaboni, Osvaldo. Risoluzione, in serie semplice, della lastra rettangolare appoggiata, sottoposta all'azione di un carico concentrato comunque disposto. Ann. Mat. Pura Appl. (4) **19**, 107–124 (1940). [MF 3047]

The problem considered is that of the bending of a simply supported rectangular plate subjected to a concentrated force applied anywhere in the interior of the plate. The solution of this problem was obtained first by Navier in the form of a double trigonometric series. Others have given solutions in certain symmetrical cases in the form of a single series which converges better and is more convenient for numerical purposes than the awkward double series. The author gives a solution in the form of a single and rapidly convergent series without assumptions of symmetry and indicates that it coincides with the solution of Navier. *J. J. Stoker* (New York, N. Y.).

Bukharinov, G. N. Solution of the two-dimensional problem of elasticity for a disc weakened by several circular holes. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **8**, 56–70 (1939). (Russian) [MF 3340]

Starting from the well-known representation

$$X_x+Y_y=4\Re[\Phi(z)],$$
$$Y_y-X_x+2iX_y=2[\bar{z}\Phi'(z)+\Psi(z)]$$

for the stresses X_x, Y_y and X_y in terms of two analytic functions Φ and Ψ of the complex variable z, the author considers the problem of the determination of Φ and Ψ for a domain consisting of points interior to a circle $|\zeta-a_0|=R_0$, and exterior to circles $|\zeta-a_k|=R_k$ $(k=1, 2, \cdots, n)$, no two circles touching, when Φ and Ψ are subjected to the boundary condition that

$$\Phi(\zeta)+\overline{\Phi}(\bar{\zeta})-e^{2i\theta}[\bar{\zeta}\Phi'(\zeta)+\Psi(\zeta)]=N-iT,$$

where ζ is a point of the boundary, N and T are the normal and tangential components of the boundary stress, and θ is the angle between the outer normal to the contour and the x-axis. Using a theorem of Harnack, the problem is reduced to a system of functional equations to which is applied a method of successive approximation similar to that used by Golusin [G. M. Golusin, Rec. Math. [Mat. Sbornik] **41**, 246–276 (1934) (Russian, with German summary)]. It is shown that the method yields a solution if the relative positions of the circles C_k $(k=0, 1, \cdots, n)$ satisfy certain conditions (in particular, that the circles are not too closely packed). *D. C. Spencer* (Cambridge, Mass.).

Lourié, A. Investigation of asymmetrical pressure of a rigid plane stamp of elliptic cross-section on an elastic semispace. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 106–109 (1940). [MF 3598]

The following problem is solved: A rigid elliptic plate supporting an eccentric vertical point load rests on the horizontal surface of a semi-infinite elastic medium. It is required to find the distribution of the stresses and displacements in the supporting medium, in particular to find the reaction pressure and the deflection under the rigid plate. All these values are expressible in terms of an harmonic function and its derivatives. By introducing elliptic coordinates the boundary conditions can be satisfied by a suitable choice of Lamé functions. From the static equivalence of the applied load and the reaction pressure, the slopes and mean deflection of the rigid plate are expressed in terms of the elastic moduli and the eccentric position of the load. In order that the reaction pressure shall be positive, the load must be applied within the middle third of the rigid plate. The vertical displacement of the center of the elliptic plate is independent of the position of the load. All the results reduce to known results when the ellipse becomes a circle. *D. L. Holl* (Ames, Iowa).

von Kármán, Th. and Tsien, Hsue-Shen. The buckling of spherical shells by external pressure. J. Aeronaut. Sci. **7**, 43–50 (1939). [MF 3277]

The purpose of the authors is to explain a discrepancy between the theoretical and experimental results for the buckling of spherical thin shells under hydrostatic pressure and to develop a new theory in better accord with experiment. The classical linear theory of buckling leads to an eigenvalue problem from which the lowest buckling pressure is calculated; this pressure is, however, considerably higher than that observed experimentally when the shell collapses. The authors develop a fundamentally new mathematical approach to the problem, based on the idea that essential elements of the problem have been lost in the linear treatment through neglect of certain non-linear terms. Assuming axial symmetry, the expression for the potential energy of the shell is derived in such a way as to include terms arising from strains in the middle surface and "large" deflections of this surface. The Euler equation obtained by variation of the potential energy expression is very cumbersome, and no attempt is made to solve it in this form. Instead, the authors introduce a simplification (which seemed justified by experimental observations) through the assumption that it is sufficient to consider only a portion of the shell determined by a small solid angle 2β. On the basis of this assumption, non-linear terms beyond a certain order may be neglected; the resulting differential equation is

$$\alpha\frac{d^2\vartheta}{d\alpha^2}+\frac{d\vartheta}{d\alpha}-\frac{\vartheta}{\alpha}=\frac{6}{(t/R)^2}\alpha\vartheta(\vartheta^2-\alpha^2)+\frac{6p}{E(t/R)^3}\alpha^2,$$

where ϑ is the inclination of the bent meridian line of the shell, α the original inclination of this line and t and R are the radius and thickness of the shell. The quantity p is the applied pressure. Even this "simplified" differential equation would be difficult to solve directly. The authors obtain an approximate solution by the Ritz method; the results,

both for the angle β and the critical load, are in quite good accord with the experimental results and are very much better than those obtained by the classical linear theory.
J. J. Stoker (New York, N. Y.).

von Kármán, Th., Dunn, Louis G. and Tsien, Hsue-Shen. The influence of curvature on the buckling characteristics of structures. J. Aeronaut. Sci. **7**, 276–289 (1940). [MF 3278]

This paper contains a qualitative discussion of buckling phenomena in a variety of cases: columns, flat plates, curved bars and spherical and cylindrical shells. The essential, and new, point of view is that most of these problems must be formulated as non-linear problems in order to bring the theoretical results in better agreement with experiment. The lines which such theories should take are indicated for the case of the spherical shell in the paper reviewed above.
J. J. Stoker (New York, N. Y.).

Biezeno, C. B. and Koch, J. J. On the buckling of a thin-walled circular tube loaded by pure bending. I. Nederl. Akad. Wetensch., Proc. **43**, 783–796 (1940). [MF 3178]

The authors investigate the elastic stability of a thin-walled, nearly cylindrical shell, acted upon by linearly distributed axial normal stresses. The initial shape of the shell is such that the deformed shell is circular and cylindrical when the stability limit is reached. The simultaneous equations determining the critical moment are of the form

$$L_1(u, v, w)=\mu\cos\phi\frac{\partial^2 u}{\partial z^2},\quad L_2(u, v, w)=\mu\cos\phi\frac{\partial^2 v}{\partial z^2},$$

$$L_3(u, v, w)=0,$$

where the L's are linear partial differential operators with constant coefficients, ϕ and z independent variables and u, v, w the three displacement components. Taking $u=\sin kzU(\phi)$, $v=\sin kzV(\phi)$, $w=\cos kzW(\phi)$, the partial equations are reduced to total equations for U, V and W. The necessarily periodic U, V, W are assumed in the form of Fourier series and the differential equations furnish a system of infinitely many homogeneous linear equations for the series coefficients. The characteristic values μ are found by equating the (symmetrical) determinant of the system equal to zero. *E. Reissner* (Cambridge, Mass.).

Biezeno, C. B. and Koch, J. J. On the buckling of a thin-walled circular tube loaded by pure bending. II. Nederl. Akad. Wetensch., Proc. **43**, 923–935 (1940). [MF 3722]

A process of iteration is described which leads, with the help of the results of the paper reviewed above, to the value of the critical bending moment. *E. Reissner.*

Riz, P. On deformations of bars with slightly curved axis. C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 110–113 (1939). [MF 2916]

The author considers the deformation of a bar, the equation of whose lateral surface has the form

$$f(x, y+(\beta z^2/2))=0,$$

where the parameter β, associated with the curvature of the axis of the bar, is regarded as small. A change of variables

$$\xi=x,\quad \eta=y+(\beta z^2/2),\quad \zeta=z$$

reduces the equation of the lateral surface to the form $f(\xi, \eta)=0$, and permits writing the equation of elasticity with ξ, η and ζ as the independent variables. The problem of flexure of a bar by couples acting at the ends of the bar is treated by a semi-inverse method with the precision up to β^2. *I. S. Sokolnikoff* (Madison, Wis.).

Riz, P. Deformations of bars with slightly curved axes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 229–232 (1939). [MF 2909]

A continuation of the paper reviewed above containing a treatment by a semi-inverse method of the torsion and the extension of a bar with slightly curved axis.
I. S. Sokolnikoff (Madison, Wis.).

Lourie, A. and Janelidze, G. On the Saint-Venant's problem for naturally twisted bars. C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 227–228 (1939). [MF 2908]

The authors report the results obtained for the stresses incurred in a naturally twisted bar by an applied flexural couple M. The angle of twist is very small so that higher powers may be neglected in the expansions employed. It is shown that the stresses may be composed of a system which is derived from a harmonic function which satisfies a relatively simple boundary condition and a second system which is formally equivalent to the combined effects of two known results, (1) the flexure due to an end force M and (2) the torsion due to a twisting couple which is readily related to the couple M. *D. L. Holl* (Ames, Iowa).

Lourie, A. and Janelidze, G. On Saint-Venant's problem for naturally twisted bars. IV. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 436–439 (1940). [MF 3221]

In three earlier notes the authors have given the general equations for the problem of a naturally twisted bar and their approximate solutions for two cases of loading: (1) flexure by a couple; (2) extension by an axial end load. In each case the nul approximation is the elementary solution obtained by simpler hypotheses. The next approximation shows that (1) is equivalent to the combined effect of a pure flexure couple together with a torsion moment applied at the end section. In (2) the same approximation shows an untwisting of the bar by the axial force. In this note the authors treat the flexure of a twisted bar by a force directed along a principal axis of inertia of the end section. The nul approximation yields the Saint Venant solution in which the shear stresses are given in terms of a harmonic function ϕ whose normal derivative is prescribed on the boundary. In the next approximation an intermediate system of stresses is explicitly determined in terms of a function conjugate to ϕ. It is further shown how this latter system may be employed to complete the problem, although no explicit results are presented. *D. L. Holl.*

Oldenburger, Rufus. Convergence of Hardy Cross's balancing process. J. Appl. Mech. **7**, A-166–A-170 (1940). [MF 3259]

The paper gives the necessary and sufficient conditions, for the case of a continuous beam, for the convergence of the Cross method and exhibits a convergent process of balancing a beam having any given number of supports. It is shown that a balancing process can be described by a real linear transformation, that is, by a matrix of real numbers and that the process converges if and only if the infinite power of this matrix exists and is zero. Reference is not made to a paper by L. Brand [Bull. Amer. Math. Soc. **41**, 901 (1935)] in which the convergence of the Cross process was investigated by another method. *H. W. March.*

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